ORIGINAL PAPER

INVERSION THEOREM FOR GENERALIZED FRACTIONAL HALF HARTLEY TRANSFORM

PRITIBALA K. SONTAKKE[1], ALKA S. GUDADHE[2]

Manuscript received: 17.07.2011 Accepted paper: 16.08.2011

Published online: 01.09.2011

Abstract. *This paper is concerned with the definition of generalized fractional half Hartley transform. Fractional half Hartley transform is extended to the distribution of compact support by using the kernel method. We establish the relation between half Hartley transform and other transforms. Lastly inversion for generalized fractional half Hartley transform is proved.*

Keywords: *Half Hartley transform, half Fourier transform, generalized fractional, half cosine and half sine transform.*

AMS subject code: *46F12 and 44*

1. INTRODUCTION

The Hartley transform arises in various areas of engineering especially in signal processing [1].

$$H\left\{f(t)\right\}(s)=\int_{-\infty}^{\infty} f(t)cas.st\ dt,\ \ t\in R^{+}. \tag{1.1}$$

Its properties such as regularity, inversion formula, etc. can be deduced from [1]. Paveri-Fontana [3] had studied half Hartley transform which arises in certain types of transport problems.

$$H\left\{f(t)\right\}(s)=\int_{0}^{\infty} f(t)cas.st\,dt, t\in R^{+}. \tag{1.2}$$

The fractional Hartley transform is defined by [2], [4] as,

[1] H.V.P.M.'S College of Engineering and Technology, 444605, Amravati, India.
E-mail: pritibala_sontakke@yahoo.co.in.
[2] Government Vidarbha Institute of Science and Humanties, Department of Mathematics, 444701, Amravati, India. E-mail: alka.gudadhe@gmail.com.

$$H^{\alpha}\{f(t)\}(s)=\int_{-\infty}^{\infty} f(t)K_{\alpha}(t,s)dt, \qquad (1.3)$$

where

$$K_{\alpha}(t,s)=\sqrt{\frac{1-i\cot\phi}{2\pi}}\,e^{i\frac{s^2}{2}\cot\phi}.e^{i\frac{t^2}{2}\cot\phi}\,\frac{1}{2}\left[(1-ie^{i\phi})cas(\csc\phi.st)+(1+ie^{i\phi})cas(-\csc\phi.st)\right].$$

Its reduced definition as in [5] is,

$$H^{\alpha}\{f(t)\}(s)=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty} e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\right]f(t)\,dt \qquad (1.4)$$

In this paper fractional half Hartley transform is defined and extended to the distribution of compact support by using the kernel method for this, testing function space is defined, in section 2. Relation between fractional half Fourier transform and fractional half Hartley transform is given in section 3. Lastly the inverse fractional half Hartley transform is obtained in section 4.

2. GENERALIZED FRACTIONAL HALF HARTLEY TRANSFORM

2.1. FRACTIONAL HALF HARTLEY TRANSFORM

The fractional half Hartley transform is defined as,

$$H^{\alpha}\{f(t)\}(s)=\int_{0}^{\infty} f(t)K_{\alpha}(t,s)dt, \qquad (2.1.1)$$

where

$$K_{\alpha}(t,s)=\sqrt{\frac{1-i\cot\phi}{2\pi}}\,e^{i\frac{s^2}{2}\cot\phi}.e^{i\frac{t^2}{2}\cot\phi}\,\frac{1}{2}\left[(1-ie^{i\phi})cas(\csc\phi.st)+(1+ie^{i\phi})cas(-\csc\phi.st)\right].$$

Equation (2.1.1) can also be simplified in the form of equation (1.4).
Now we extend it to the generalized function.

2.2. THE TEST FUNCTION SPACE $E(R^n)$

An infinitely differentiable complex valued function ψ on R^n belongs to $E(R^n)$ for each compact set $K \subset S_a$ where $S_a=\{t \in R^n, |t| \le a,\ a>0\}$,

$$\gamma_K(\psi) = \sup_{t \in K} \left| D_t^k \, \psi(t) \right| < \infty, \quad \text{k=1, 2, 3…}$$

Note that the space *E* is complete and therefore a Frechet space.

2.3. THE FRACTIONAL HARTLEY TRANSFORM ON E'

It can be easily proved that function $K_\alpha(t,s)$ as a function of t, is a member of $E(R^n)$, where

$$K_\alpha(t,s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} . e^{i\frac{t^2}{2}\cot\phi}\, \frac{1}{2}\left[(1 - ie^{i\phi}) cas(\csc\phi.st) + (1 + ie^{i\phi}) cas(-\csc\phi.st)\right],$$

since $\phi = \frac{\alpha\pi}{2}$ then $K_\alpha(t,s) \in E(R^n)$ if ,

$$\gamma_K \left| K_\alpha(t,s) \right| = \sup_{0<t<\infty} \left| D_t^k K_\alpha(t,s) \right| < \infty \, .$$

The generalized fractional half Hartley transform of $f(t) \in E'(R^n)$, where $E'(R^n)$ is the dual of the testing function space, can be defined as,

$$H^\alpha \{ f(t) \}(s) = \langle f(t), K_\alpha(t,s) \rangle. \qquad (2.2.1)$$

3. RELATION BETWEEN DIFFERENT TRANSFORMS

3.1. FRACTIONAL HALF HARTLEY TRANSFORM AND HALF HARTLEY TRANSFORM

As fractional half Hartley transform is given by

$$H^\alpha \{ f(t) \}(s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} \int_0^\infty e^{i\frac{t^2}{2}\cot\phi} \left[\cos(\csc\phi.st) - ie^{i\phi}\sin(\csc\phi.st)\right] f(t)\, dt \, ,$$

where $\phi = \alpha \frac{\pi}{2}$.

It can be easily seen that by putting $\alpha = 1$, $\phi = \frac{\pi}{2}$ fractional half Hartley transform becomes,

$$H\{ f(t) \}(s) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f(t) cas.st \, dt \, .$$

This is conventional half Hartley transform. Half range fractional Hartley transform is also called one-sided fractional Hartley transform, since ϕ in between 0 to ∞.

3.2. RELATION BETWEEN FRACTIONAL HALF FOURIER TRANSFORM AND FRACTIONAL HALF HARTLEY TRANSFORM

The fractional half Fourier transform with angle α of a signal $f(t)$ defined as,

$$F_\alpha\{f(t)\}(s)=\sqrt{\frac{1-i\cot\phi}{2\pi}}\int_0^\infty f(t)e^{\frac{i\cot\phi}{2}(t^2+s^2-2i\csc\phi.st)}dt ,$$

The relation between fractional half Fourier transform and fractional half Hartley transform is given by,

$$F_\alpha\{f(t)\}(\mathrm{s})=\frac{1}{2}\left[(1+e^{-i\phi})H^\alpha\{f(t)\}(s)+(1-e^{-i\phi})H^\alpha\{f(t)\}(-s)\right] \tag{3.2.1}$$

This can be easily proved.

4. INVERSION OF THE FRACTIONAL HALF HARTLEY TRANSFORM

4.1. STATEMENT

Let $f\in E^{'}(R)$ and let its fractional half Hartley transform be defined by (2.1.1), it is possible to recover the function f by means of the inversion formula.

$$f(t)=\frac{1}{1+e^{2i\phi}}\sqrt{\frac{2\pi}{1-i\cot\phi}}e^{-i\frac{t^2}{2}\cot\phi}\int_0^\infty e^{-i\frac{s^2}{2}\cot\phi}H^\alpha\{f(t)\}(s)\left[\cos(\csc\phi.st)+ie^{i\phi}\sin(\csc\phi.st)\right]ds .$$

Proof: Fractional half Hartley transform be defined as,

$$H^\alpha\{f(t)\}(s)=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_0^\infty e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\right]f(t)\,dt$$

That is

$$\sqrt{\frac{2\pi}{1-i\cot\phi}}\,e^{-i\frac{s^2}{2}\cot\phi}H^{\alpha}\{f(t)\}(s)$$
$$=\int_0^\infty e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)\right]f(t)\,dt-ie^{i\phi}\int_0^\infty e^{i\frac{t^2}{2}\cot\phi}\sin(\csc\phi.st)\,f(t)\,dt. \tag{4.1.1}$$

Using inverse half cosine transform to both side of equation (4.1.1), we get

$$\sqrt{\frac{2\pi}{1-i\cot\phi}}\int_0^\infty e^{-i\frac{s^2}{2}\cot\phi}H^{\alpha}\{f(t)\}(s)\cos(\csc\phi.st)\,ds$$
$$=e^{i\frac{t^2}{2}\cot\phi}f(t)-ie^{i\phi}\int_0^\infty\int_0^\infty e^{i\frac{t^2}{2}\cot\phi}\sin(\csc\phi.st)\cos(\csc\phi.st)\,f(t)\,dtds. \tag{4.1.2}$$

Also, using inverse half sine transform to both side of equation (4.1.1), it becomes

$$\sqrt{\frac{2\pi}{1-i\cot\phi}}\int_0^\infty e^{-i\frac{s^2}{2}\cot\phi}H^{\alpha}\{f(t)\}(s)\sin(\csc\phi.st)\,ds$$
$$=\int_0^\infty\int_0^\infty e^{i\frac{t^2}{2}\cot\phi}\cos(\csc\phi.st)\sin(\csc\phi.st)\,f(t)\,dtds-ie^{i\phi}e^{i\frac{t^2}{2}\cot\phi}f(t). \tag{4.1.3}$$

Now multiplying equation (4.1.3) by $ie^{i\phi}$ and adding with (4.1.2), we get

$$f(t)=\frac{1}{1+e^{2i\phi}}\sqrt{\frac{2\pi}{1-i\cot\phi}}\,e^{-i\frac{t^2}{2}\cot\phi}\int_0^\infty e^{-i\frac{s^2}{2}\cot\phi}H^{\alpha}\{f(t)\}(s)\left[\cos(\csc\phi.st)+ie^{i\phi}\sin(\csc\phi.st)\right]ds.$$

This is the required inverse of the fractional half Hartley transform.

5. CONCLUSIONS

In this paper fractional half Hartley transform is extended to the distribution of compact support by using the kernel method. Relation between fractional half Hartley transform and half Hartley transform is given. Relation between fractional half Fourier transform and fractional half Hartley transform is also established. Lastly inversion of fractional half Hartley transform is proved.

Since fractional Hartley transform is useful signal processing tool, we believe that fractional half Hartley transform will also become useful tool for signal processing and image processing.

REFERENCES

[1] Bracewell, R.N., *The Hartley Transform*, Oxford U. K. Oxford Univ. Press, 1986.
[2] Namias, V., *J Inst. Math Apptics*, **25**, 241, 1980.
[3] Paveri-Fontana, S.L, Zweifel P.F., *J. Math. Phy.*, **35**(5), 2648, 1994.
[4] Pei, S.C., Ding J. J., *IEEE, trans. on signal processing*, **50**(7), 2002.
[5] Sontakke, P.K., Gudadhe, A.S., *Vidarbha Journal of Science*, **II**(1), 2007.
[6] Zemanian, A.H., *Distribution Theory and Transform Analysis*, Mc.Graw Hill, NewYork, 1965.
[7] Zemanian, A.H., *Generalized integral transform*, Inter Science Publisher's, NewYork, 1968.