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OPTIMALITY IN NONDIFFERENTIAL MULTIOBJECTIVE PROGRAMMING

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Abstract. *This paper is concerning on the multiobjective programming problem where the function involved are nondifferentiable. We present and prove the sufficient conditions for a feasible point to be weakly efficient. Our research starts from the invexity proposed by H. Slimani and M.S. Rajdef and extend their concept for the case when the functions are nondifferentiable. To solve the problem in this new framework we show how local cone approximation concept can be used. Thus we provide a new approach for the nondifferentiable multiobjective programming problems that can be easily applied in the practical problems.*

Keywords: *Multiobjective programming, K-directional derivative.*

AMS subject code: *90C29.*

1. INTRODUCTION

The multiobjective programming represents an optimization situation and has many of utilizations in the practice. For example, there are a great number of economical problems where the multiobjective programming is applied (see [1]). In the last decade the scientists started to develop this research field and gave their contributions to design new models which describe and solve the multiobjective programming problems.

For I a nonempty open subset of R^n and for two vector functions $f : I \to R^p$ and $g : I \to R^m$ a constrained multiobjective programming problem (VP) is an optimization problem as follows:

$$\min \ f(x) ,$$
$$g(x) \le 0, \quad x \in I$$

where $f_i : I \to R$, $g_i : I \to R$ with i=1,p and j=1,m.

The set $X = \{x \in I \quad with \quad g(x) \le 0\}$ is named the feasible solutions set of (VP). For x_0 , we note with $J(x_0)$ the set: $J(x_0) = \{ j = 1, m \quad by \quad g_j(x_0) = 0 \}$.

Definition 1. A feasible solution $x_0 \in X$ is said to be an weakly efficient solution of (VP) if there is no $x \in X$ such that $f(x) < f(x_0)$.

Definition 2. A feasible solution $x_0 \in X$ is said to be an efficient solution of (VP) if there is no $x \in X$ such that $f(x) \le f(x_0)$, $f(x) \ne f(x_0)$.

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Definition 3. A feasible solution $x_0 \in X$ is said to be a properly efficient solution of (VP) if it is efficient and there exists a positive constant a such that for each $x \in X$ and for each i=1,..,p satisfying $f_i(x) < f_i(x_0)$, there exists at least one j=1,..,p such that $f_j(x_0) < f_j(x)$ and $f_i(x_0) - f_i(x) \le a(f_j(x) - f_j(x_0))$.

To establish sufficient optimality conditions and to study the duality theorems for a multiobjective programming problem have to taking in account an important aspect, namely the convexity. The class of convex functions was considered as a restrictive class and to weak the limitations of the convex functions class was defined several new functions classes. These classes are a generalization of convex functions class.

Hanson (see [5]) was one that introduced the class of invex functions. This class of functions intends to relax the convexity assumptions that must be imposed on functions to establish sufficient optimality conditions. This research domain was faster developed and the specialized literature contains now several new concepts concerning the invex functions class [4, 7]. Our paper starts from the H. Slimani and M.S. Radjef approach. They proposed studying the invexity of a differentiable vector function by considering the invexity of each component with respect to its own function η_i .

In the following exposition we study the optimality conditions for the multiobjective programming problems where the functions involved are nondifferentiable. There is some approaches that use K-directional derivative where K is a local cone approximation [2, 6],. In this paper we generalize the invexity concept. We state the definition of ρ-invexity for nonsmooth functions with respect to different η_i, using the concept of local cone approximation.

The local cone approximation was introduced by Elster and Thierfelder in [3] and it allows us to determine the directional derivative for a function f at x.

The set epigraf of f for a function $f : I \to R$ where I is a nonempty open subset of R^n, is:

$$epi \quad f = \{(x,a) \in I \times R \quad with \quad f(x) \le a\}$$

The locally approximation of the set $epi\ f$ at the point $(x, f(x))$ define the local cone approximation K and for a gave K we can uniquely determine the K-directional derivative of f at x.

Definition 4. If $f : I \to R$ let $x \in I$ and K the local cone approximation. The positively homogeneous function $f^K(x,.) : R^n \to [-\infty, \infty]$ that is defined by:

$$f^K(x,d) = \boldsymbol{inf}\left\{\xi \in R \quad with(d,\xi) \in K\left(epi\ f,(x, f(x))\right)\right\}$$

is called the K- directional derivative of f at x.

In Section 2 we define $\rho-$invex and (weakly) $\rho-$pseudo-invex vector functions with respect to different $(\eta_i)_{i=1,p}$ using K-directional derivative of a function f at x. In Sections 3 we present a theorem that establish sufficient conditions for a feasible point to be weakly efficient when all the function $(\eta_i)_{i=1,p}$ are equal.

2. DEFINITIONS

In this section we give definitions for $\rho-$invexity and (weakly) $\rho-$pseudo-invexity considering invexity with respect to different $(\eta_i)_{i=\overline{1,p}}$. We work with nondifferentiable functions.

If I is a nonempty open set of R^n and f is a nondifferentiable function $f: I \to R^p$, then we have:

Definition 5. Let a nondifferentiable function $f: I \to R^p$ and K be a local cone approximation. We said that the function f is strictly K-ρ-invex at $x_0 \in I$ with respect to $(\eta_i)_{i=\overline{1,p}}$, if there exist N vector function $\eta_i : I \times I \to R^n$, $i=\overline{1,p}$ and $\rho = (\rho_1,...,\rho_p)$ a vector with real components where $\rho_i \in R$, $i=\overline{1,p}$ such that for each $x \in I$ we get:

$$f_i(x) - f_i(x_0) > f_i^K(x_0, \eta_i(x,x_0)) + \rho_i d(x,x_0) \text{ for all } i=\overline{1,p}.$$

The function f is named to be strictly K-ρ-invex on I with respect to $(\eta_i)_{i=\overline{1,p}}$, if is strictly K-$\rho$-invex at each $x_0 \in I$ with respect to the same $(\eta_i)_{i=\overline{1,p}}$.

Definition 6. Let a nondifferentiable function $f: I \to R^p$ and K a local cone approximation. We said that the function f is strictly K-ρ-weakly pseudo-invex at $x_0 \in I$ with respect to $(\eta_i)_{i=\overline{1,p}}$, if there exist p vector functions $\eta_i : I \times I \to R^n$, $i=\overline{1,p}$ and $\rho = (\rho_1,...,\rho_p)$ a vector with real components such that for a $x \in I$ with $f(x) - f(x_0) < 0$ exists $\overline{x} \in I$ such that:

$$f_i^K(x_0, \eta_i(\overline{x}, x_0)) + \rho_i d(\overline{x}, x_0) < 0 \text{ for all } i=\overline{1,p}.$$

For $\overline{x} = x$ the definition 6 becomes the definition for f a K-ρ-weakly pseudo-invex at $x_0 \in I$ with respect to $(\eta_i)_{i=\overline{1,p}}$.

If f is (weakly) K-ρ-pseudo-invex at each $x_0 \in I$ with respect to the same $(\eta_i)_{i=\overline{1,p}}$ then the function f is (weakly) K-ρ-weakly pseudo-invex on I with respect to $(\eta_i)_{i=\overline{1,p}}$.

Definition 7. A nondifferentiable function $f: I \subseteq R^n \to R$ for $\sigma \in R,\ \ \sigma \geq 0$, is named to be K-$\sigma$ -quasi-invex in x_0 with respect to $\theta: I \times I \to R^n$ if there exists a vector function θ such that:

$$h(x) - h(x_0) \leq 0 \Rightarrow h_j^K(x_0, \theta_j(x,x_0)) + \sigma d(x,x_0) \leq 0 \quad \forall x \in I$$

3. THE SUFFICIENT CONDITIONS FOR A FEASIBLE POINT TO BE WEAKLY EFFICIENT

In the following exposition we present the sufficient conditions for a feasible point to be weakly efficient for (VP). We study a certain case when all function $(\eta_i)_{i=\overline{1,p}}$ and θ_j, $j \in J(x_0)$ are equal and we noted these functions with η. Our results are organized in a theorem which supplies the conditions for which $x_0 \in X$ is a weakly efficient solution for a (VP).

Theorem 1 Let $x_0 \in X$ and supposes that:

i_1 f is K-ρ-weakly pseudo-invex at $x_0 \in I$ with respect to $\eta : X \times X \to R^n$;

i_2 If

$$g_j^K\left(x_0, \eta(x, x_0)\right) \le -\sigma_j d(x, x_0),$$
$$\forall x \in X \ , \ j \in J(x_0) \text{ and for vector } \sigma = (\sigma_j)_{j \in J(x_0)} \ ;$$

i_3 There exist vectors $\mu \in R^p, \mu \ge 0$ and $\lambda \in R^{cardJ(x_0)}, \lambda \ge 0$ such that $(x_0, \mu, \lambda, \eta)$ satisfy the following conditions:

$$\sum_{i=1}^{p} \mu_i f_i^K\left(x_0, \eta(x, x_0)\right) + \sum_{j \in J(x_0)} \lambda_j g_j^K\left(x_0, \eta(x, x_0)\right) = 0 \quad \forall x \in X$$

i_4

$$\sum_{i=1,..,p} \rho_i \mu_i + \sum_{j \in J(x_0)} \lambda_j \sigma_j \ge 0$$

then x_0 is a weakly efficient solution for (VP).

Proof: We start by the false hypothesis that x_0 is not a weakly efficient solution for (VP). In this situation we have a x for that $f(x) - f(x_0) < 0$. From the hypothesis i_1 , we have that the function f is K-ρ-weakly pseudo-invex at x_0 with respect to η . That means:

$$\exists \quad \bar{x} \in I \quad with$$
$$f_i^K\left(x_0, \eta(\bar{x}, x_0)\right) < -\rho_i d(\bar{x}, x_0) \quad \forall \quad i = 1,.., p \tag{1}$$

If we make the summation by the index i , the relation (1) becomes:

$$\sum_{i=1}^{p} f_i^K\left(x_0, \eta(\bar{x}, x_0)\right) < -\sum_{i=1}^{p} \rho_i d(\bar{x}, x_0) \tag{2}$$

And much more, if we multiply each term of the sums with the components of the vector $\mu \ge 0$, we get:

$$\sum_{i=1}^{p} \mu_i f_i^K\left(x_0, \eta(\bar{x}, x_0)\right) < -\sum_{i=1}^{p} \rho_i \mu_i d(\bar{x}, x_0) \tag{3}$$

On the other hand, the hypothesis i_2 supplies:

$$g_j^K\left(x_0, \eta(\bar{x}, x_0)\right) \le -\sigma_j d(\bar{x}, x_0) \tag{4}$$

The previous relation implies:

$$\sum_{j \in J(x_0)} g_j^K\left(x_0, \eta(\bar{x}, x_0)\right) \le -\sum_{j \in J(x_0)} \sigma_j d(\bar{x}, x_0) \tag{5}$$

Taking in to account that $\lambda \ge 0$, we have:

$$\sum_{j \in J(x_0)} \lambda_j g_j^K\left(x_0, \eta(\bar{x}, x_0)\right) \le -\sum_{j \in J(x_0)} \lambda_j \sigma_j d(\bar{x}, x_0) \tag{6}$$

The relations (3) and (6) imply that:

$$\sum_{i=1}^{p}\mu_i f_i^K\left(x_0,\eta\left(\bar{x},x_0\right)\right)+\sum_{j\in J(x_0)}\lambda_j g_j^K\left(x_0,\eta\left(\bar{x},x_0\right)\right)<-\left(\sum_{i=1}^{p}\rho_i\mu_i+\sum_{j\in J(x_0)}\lambda_j\sigma_j\right)d\left(\bar{x},x_0\right) \qquad (7)$$

Using the hypothesis i_4 :

$$\sum_{i=1}^{p}\rho_i\mu_i+\sum_{j\in J(x_0)}\lambda_j\sigma_j\geq 0$$

We obtain that the right member of the relation (7) is a negative real number and so the left member of the same relation becomes a strictly negative one:

$$\sum_{i=1}^{p}\mu_i f_i^K\left(x_0,\eta_i\left(\bar{x},x_0\right)\right)+\sum_{j\in J(x_0)}\lambda_j g_j^K\left(x_0,\theta_j\left(\bar{x},x_0\right)\right)<0 \qquad (8)$$

Now we observe that the relation (8) is in contradiction with the hypothesis i_3 that said:

$$\sum_{i=1}^{p}\mu_i f_i^K\left(x_0,\eta\left(x,x_0\right)\right)+\sum_{j\in J(x_0)}\lambda_j g_j^K\left(x_0,\eta\left(x,x_0\right)\right)=0 \quad \forall x\in X$$

So our initial hypothesis that x_0 is not a weakly efficient solution for (VP) is false and x_0 is a weakly efficient solution for (VP).

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