***ORIGINAL PAPER***

# A NOTE ON RINGS

VOLETY V.S. RAMACHANDRAM[1]

*Manuscript received: 22.08.2011; Accepted paper: 30.08.2011*

*Published online: 01.09.2011*

***Abstract.*** *In this paper we have mainly obtained some properties satisfied by the invertible and complement elements of a ring R with unity. We obtained that the complement of an idempotent element in a ring is idempotent. We obtained a necessary and sufficient condition for an element in a ring R to be invertible.*

***Keywords***: *Ring, Ring with unity, Boolean Ring, complement of an element, invertible element and idempotent element.*

***AMS subject classification***: *13XX.*

We start with the following Preliminaries:

**Definition 1**: A ring R is a Boolean ring if $a^2 = a \quad \forall a \in R$.

**Definition 2**: In a ring R with unity an element 'a' has complement [2] if there exist an element 'b' in R such that a + b =1.

**Definition 3**: In a commutative ring R with unity an element 'a' is invertible [2] if there exist an element 'b' in R such that a. b = 1.

First we start with the following theorem.

**Theorem 1**: In a ring (R, +, .) with unity, $a \in R$ has complement if 'a' is invertible.
*Proof*: Suppose $a \in R$ is invertible. There exists $b \in R$ such that $a.b = b.a = 1$
$a + a.(b-1) = a + a.b - a.1 = a + 1 - a = 1$, which proves the theorem.

**Remark**: The converse of the above theorem need not be true.
For example in the ring of integers (Z, +, .) every element has complement but some elements have no inverse.

**Theorem 2**: In a ring (R, +, .) with unity, the complement of an idempotent element in R is an idempotent element in R.
*Proof*: Let 'a' be an idempotent element in R and 'b' be its complement.

$$b.b = (1-a).(1-a) = 1-a-a+a.a = 1-a-a+a = 1-a = b$$

**Theorem 3**: In a Boolean ring [1] with unity, $a + b = 1 \Rightarrow a.b = 0$
*Proof*: Let R be a Boolean ring with unity.

$$a + b = 1 \Rightarrow b = 1-a$$
$$\Rightarrow a.b = a.(1-a) \Rightarrow a.b = a - a.a \Rightarrow a.b = a - a \Rightarrow a.b = 0$$

[1] B.V.C. College of Engineering, Rajahmundry. 533210, Andhra Pradesh, India. E-mail: vvsrch@yahoo.com.

**Theorem 4**: In a commutative ring R with unity the following conditions are equivalent.

$a \in R$ is invertible

The equations a.x = b and y.a = b have unique solution $\forall x, y, b \in R$

*Proof*: Suppose (1) hold.

There is an element $i \in R$ such that $a.i = i.a = 1$.

Consider the equation $ax = b$

$$\Rightarrow i.(ax) = i.b \Rightarrow (i.a).x = i.b \Rightarrow 1.x = i.b \Rightarrow x = i.b$$

Hence the equation a.x = b has a solution in the ring R.

If possible $a.r = a.s$ for some r, s in R.

$$a.r = a.s \Rightarrow i.(a.r) = i.(a.s) \Rightarrow (i.a).r = (i.a).s \Rightarrow r = s$$

Suppose (2) hold.

The equation a.x = b has unique solution means there is a unique element say ‘q’ in R such that a .q =1 so that ‘q’ is the inverse of ‘a’ in R.

**Lemma 5**: In a ring R with unity if ‘a’ and ‘b’ are invertible in R then $(a.b)^{-1} = b^{-1}.a^{-1}$

*Proof*: $a.b.(b^{-1}a^{-1}) = a.(bb^{-1}).a^{-1} = a.1.a^{-1} = a.a^{-1} = 1$ .

**Theorem 6**: If $R$ and $R'$ are two rings with unity elements, f is a homomorphism from $R$ to $R'$ then the following are true:

‘a’ is idempotent in R then f(a) is idempotent in $R'$.

‘a’ is invertible in R then f(a) is invertible in $R'$.

‘a’ has complement in R then f(a) has complement in $R'$.

*Proof*: It is trivial.

**Theorem 7**: In a commutative ring R with unity the inverse of an idempotent element (if exists) is idempotent.

*Proof*: Let ‘a’ be an idempotent element in the ring R and ‘b’ be its inverse.

$$a.b = 1 \Rightarrow (a.b)^2 = 1 \Rightarrow a^2.b^2 = 1 \Rightarrow a.b^2 = 1 \Rightarrow b.a.b^2 = b \Rightarrow 1.b^2 = b \Rightarrow b^2 = b$$

Hence ‘b’ is idempotent in R.

## REFERENCES


[1] Herstein, I.N., *Topics in Algebra*, Wiley India (P) Ltd, 2nd Edition, 2006.

[2] Srinivas, K.V.R., Ramachandram, V.V.S., *International Journal of Mathematics Research*, **3**(1), 53, 2011.

[3] Ashraf, M., Quadri, M.A., Zelinsky, D., *Amer. Math.Monthly*, **95**(4), 336, 1988.

[4] Gupta, R.N., *Math. Student*, **39**, 184, 1971.

[5] Yuanchun, G., *Acta Scientiarum Naturalium Universitatis Jilinensis*, **3**, 11, 1983.