ORIGINAL PAPER

ON AN INEQUALITY OF OSTROWSKI TYPE

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Abstract. *The main purpose of this paper is to derive new inequalities of Ostrowski type in the weighted case. New estimations of the remainder term in quadrature formulas are obtained. Also, the two-point Ostrowski inequality is considered.*

Keywords: *Ostrowski inequality, Two-point Ostrowski inequality, Mean value theorem.*

1. INTRODUCTION

In the last years the inequalities of Ostrowski type have occupied the attention of many authors ([2-5, 8, 9, 11]).

In [4], S.S. Dragomir using mean value theorems proved the following inequality of Ostrowski type

Theorem 1. ([4]) *Let* $f:[a,b]\to R$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $[a,b]$ *not containing* 0. *Then for any* $x\in[a,b]$, *we have the inequality*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\frac{b-a}{|x|}\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right]\cdot\|f-lf'\|_\infty, \tag{1}$$

where $l(t)=t,\ t\in[a,b]$.

In [10], E.C. Popa using a mean value theorem obtained a generalization of Dragomir's result.

Theorem 2. ([10]) *Let* $f:[a,b]\to R$ *be continuous on* $[a,b]$ *and differentiable on (a,b). Then for any* $x\in[a,b]$ *we have the inequality*

$$\left|\left[\frac{a+b}{2}-\alpha\right]f(x)+\frac{\alpha-x}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)\|f-lf'\|_\infty,$$

where $\alpha\notin[a,b]$ *and* $l(t)=t-\alpha,\ t\in[a,b]$.

In [8], J. Pečarić and S. Ungar have proved a general estimate with the p-norm, $1\le p\le\infty$ which for $p=\infty$ give the Dragomir's result.

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Theorem 3. ([8]) *Let the function* $f:[a,b]\to R$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0<a<b$. *Then for* $\frac{1}{p}+\frac{1}{q}=1$, *with* $1\le p,q\le\infty$, *and all* $x\in[a,b]$, *the following inequality holds*

$$\left|\frac{a+b}{2}\cdot\frac{f(x)}{x}-\frac{1}{b-a}\int_a^b f(t)dt\right|\le PU(x,p)\cdot\left\|f-lf'\right\|_p, \qquad (2)$$

where $l(t)=t,\ t\in[a,b]$, *and*

$$PU(x,p)=(b-a)^{\frac{1}{p}-1}\left[\left(\frac{a^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-a^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}+\left(\frac{b^{2-q}-x^{2-q}}{(1-2q)(2-q)}+\frac{x^{2-q}-b^{1+q}x^{1-2q}}{(1-2q)(1+q)}\right)^{\frac{1}{q}}\right].$$

Note that in cases $(p,q)=(1,\infty)$, $(\infty,1)$ and $(2,2)$ the constant $PU(x,p)$ has to be taken as the limit as $p\to 1,\infty$ and 2, respectively.

The main purpose of this paper is to derive new inequalities of Ostrowski type, generalizing some results of S.S. Dragomir, J. Pečarić, S. Ungar and E.C. Popa (see [4], [8], [10]).

2. THE INEQUALITY OF OSTROWSKI TYPE

The following result is a generalization of the Ostrowski type inequalities, which were enumerated in the first section.

Theorem 4. *Let the functions* $f,w:[a,b]\to R$ *be contiuous on* $[a,b]$ *and differentiable on* (a,b) *and* $w(x)\ne 0$ *for all* $x\in[a,b]$. *Then for* $\frac{1}{p}+\frac{1}{q}=1$, *with* $1\le p,q\le\infty$, *and all* $x\in[a,b]$, *the following inequality holds:*

$$\left|\frac{f(x)}{w(x)}\int_a^b w(t)dt-\int_a^b f(t)dt\right|\le K(x,p)\cdot\left\|w'f-wf'\right\|_p,$$

where

$$K(x,p)=(b-a)^{\frac{1}{p}}\left\{\left[\int_a^x\left(\int_t^x\frac{|w(t)|^q}{w(u)^{2q}}du\right)dt\right]^{\frac{1}{q}}+\left[\int_x^b\left(\int_x^t\frac{|w(t)|^q}{w(u)^{2q}}du\right)dt\right]^{\frac{1}{q}}\right\}.$$

Proof. Define $F:[a,b]\to R$, $F(x)=\frac{f(x)}{w(x)}$. The function F is continuous and differentiable on (a,b), and for all $x,t\in[a,b]$ we have

$$F(x)-F(t)=\int_t^x F'(u)du=\int_t^x\frac{1}{w(u)^2}\left[f'(u)w(u)-f(u)w'(u)\right]du.$$

Therefore

$$\frac{f(x)}{w(x)}\int_a^b w(t)dt-\int_a^b f(t)dt=\int_a^b w(t)\left[\int_x^t \frac{1}{w(u)^2}\big(w'(u)f(u)-w(u)f'(u)\big)du\right]dt.$$

We have

$$\left|\frac{f(x)}{w(x)}\int_a^b w(t)dt-\int_a^b f(t)dt\right|\le\int_a^b\left|\int_x^t\left|\big(w'(u)f(u)-w(u)f'(u)\big)\frac{w(t)}{w(u)^2}\right|du\right|dt=$$

$$\int_a^x\left(\int_t^x\left|\big(w'(u)f(u)-w(u)f'(u)\big)\frac{w(t)}{w(u)^2}\right|du\right)dt+\int_x^b\left(\int_x^t\left|\big(w'(u)f(u)-w(u)f'(u)\big)\frac{w(t)}{w(u)^2}\right|du\right)dt\le$$

$$\left[\int_a^x\left(\int_t^x\left|w'(u)f(u)-w(u)f'(u)\right|^p du\right)dt\right]^{\frac{1}{p}}\left[\int_a^x\left(\int_t^x\frac{|w(t)|^q}{w(u)^{2q}}du\right)dt\right]^{\frac{1}{q}}+$$

$$\left[\int_x^b\left(\int_x^t\left|w'(u)f(u)-w(u)f'(u)\right|^p du\right)dt\right]^{\frac{1}{p}}\left[\int_x^b\left(\int_x^t\frac{|w(t)|^q}{w(u)^{2q}}du\right)dt\right]^{\frac{1}{q}}\le$$

$$\left[\int_a^b\left(\int_a^b\left|w'(u)f(u)-w(u)f'(u)\right|^p du\right)dt\right]^{\frac{1}{p}}\left\{\left[\int_a^x\left(\int_t^x\frac{|w(t)|^q}{w(u)^{2q}}du\right)dt\right]^{\frac{1}{q}}+\left[\int_x^b\left(\int_x^t\frac{|w(t)|^q}{w(u)^{2q}}du\right)dt\right]^{\frac{1}{q}}\right\}\le$$

$$K(x,p)\cdot\left\|w'f-wf'\right\|_p.$$

Remark 1. *For* $w(t)=t$, $t\in[a,b]$, $0<a<b$, *we obtain the Pečarić and Ungar's result and for* $w(t)=t-\alpha$, $t\in[a,b]$, *where* $\alpha\notin[a,b]$ *and* $p=\infty$, *we obtain E.C. Popa's result (see [10]).*

Remark 2. *For* $w(t)=1$, $t\in[a,b]$, *we obtain the following Ostrowski inequality*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le(b-a)^{\frac{1}{p}-1}\left\{\left[\frac{(x-a)^2}{2}\right]^{\frac{1}{q}}+\left[\frac{(b-x)^2}{2}\right]^{\frac{1}{q}}\right\}\left\|f'\right\|_p.$$

If we consider $p=\infty$ *and* $|f'(t)|\le M$ *for all* $t\in(a,b)$, *the classical Ostrowski's inequality ([7]) is recaptured*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)dt\right|\le\left[\frac{1}{4}+\left(\frac{x-\frac{a+b}{2}}{b-a}\right)^2\right](b-a)M.$$

3. ON THE TWO-POINT OSTROWSKI INEQUALITY

Let $f:[a,b]\to R$ be a function satisfying the Lipschitz condition with constant $M>0$ and $a\le c<d\le b$. In [5] Matić and Pečarić proved the following two-point Ostrowski inequality:

$$\left|\frac{1}{b-a}\int_a^b f(t)dt-\frac{1}{d-c}\int_c^d f(x)dx\right|\le\frac{(c-a)^2+(b-d)^2}{2(c-a+b-d)}M.$$

This result was generalized in [1, 6, 9]. We recall here the generalization obtained by M. Matić and S. Ungar in [6].

Theorem 5. *([6]) Let the function* $f:[a,b]\to R$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *with* $0<a<b$. *Then for* $\frac{1}{p}+\frac{1}{q}=1$, *with* $1\le p,q\le\infty$, *and numbers* $a\le c<d\le b$, *the following inequality holds:*

$$\left|(b^2-a^2)\int_c^d f(x)dx-(d^2-c^2)\int_a^b f(t)dt\right|\le\frac{1}{2}\|K\|_q\cdot\|f-lf'\|_p$$

where

$$K(u)=\begin{cases}(d^2-c^2)\left(\dfrac{a^2}{u^2}-1\right), & a\le u\le c,\\ b^2-a^2+c^2-d^2-\dfrac{b^2c^2-a^2d^2}{u^2}, & c\le u\le d,\\ (d^2-c^2)\left(\dfrac{b^2}{u^2}-1\right), & d\le u\le b.\end{cases}$$

In this section, in a similar manner, we will obtain an estimate of the two-point Ostrowski type, which in a special case reduce to M. Matić and S. Ungar's result from [6].

Lemma 1. *Let the functions* $f,w:[a,b]\to R$ *be continuous on* $[a,b]$ *and differentiable on* (a,b) *and* $w(x)\ne 0$ *for all* $x\in[a,b]$. *Then*

$$w(t)f(x)-w(x)f(t)=w(t)w(x)\int_x^t\left[f(u)w'(u)-f'(u)w(u)\right]\frac{1}{w(u)^2}du.$$

Theorem 6. *Let the functions* $f,w:[a,b]\to R$ *be continuous on* $[a,b]$ *and differentable on* (a,b) *and* $w(x)\ne 0$ *for all* $x\in[a,b]$. *Then for* $\frac{1}{p}+\frac{1}{q}=1$, *with* $1\le p,q\le\infty$ *and numbers* $a\le c<d\le b$, *the following inequality holds*

(10) $\left|\int_c^d f(x)dx\int_a^b w(t)dt-\int_c^d w(x)dx\int_a^b f(t)dt\right|\le\|K\|_q\|G\|_p$, *where*

$$K(u)=\begin{cases}\dfrac{1}{w(u)^2}D(d,c)D(a,u), & a\le u\le c,\\ \dfrac{1}{w(u)^2}\left(D(c,u)D(d,b)-D(u,d)D(a,c)\right), & c\le u\le d,\\ \dfrac{1}{w(u)^2}D(c,d)D(u,b), & d\le u\le b,\end{cases}$$

$$D(y,z)=\int_y^z w(t)dt,\quad y,z\in[a,b],\ \text{and}\ G(t)=f(t)w'(t)-f'(t)w(t),\ \ t\in[a,b].$$

Proof. Applying Lemma 1 for the function f and integrating on t over $[a,b]$, gives

$$f(x)\int_a^b w(t)dt - w(x)\int_a^b f(t)dt = w(x)\int_a^b\left(w(t)\int_x^t G(u)\frac{1}{w(u)^2}du\right)dt =$$

$$-\int_a^x\left(\int_a^u G(u)\frac{w(x)w(t)}{w(u)^2}dt\right)du + \int_x^b\left(\int_u^b G(u)\frac{w(x)w(t)}{w(u)^2}dt\right)du =$$

$$-w(x)\int_a^x \frac{D(a,u)}{w(u)^2}G(u)du + w(x)\int_x^b \frac{D(u,b)}{w(u)^2}G(u)du.$$

Integrating this identity on x over $[c,d]$, gives

$$\int_c^d f(x)dx\int_a^b w(t)dt - \int_c^d w(x)dx\int_a^b f(t)dt =$$

$$-\int_c^d w(x)\left(\int_a^x \frac{D(a,u)}{w(u)^2}G(u)du\right)dx + \int_c^d w(x)\left(\int_x^b \frac{D(u,b)}{w(u)^2}G(u)du\right)dx =$$

$$\int_c^a\left(\int_c^u \frac{w(x)}{w(u)^2}D(a,u)G(u)dx\right)du - \int_a^d\left(\int_u^d \frac{w(x)}{w(u)^2}D(a,u)G(u)dx\right)du +$$

$$\int_c^b\left(\int_c^u \frac{w(x)}{w(u)^2}D(u,b)G(u)dx\right)du - \int_b^d\left(\int_u^d \frac{w(x)}{w(u)^2}D(u,b)G(u)dx\right)du =$$

$$\int_a^c D(d,c)D(a,u)\frac{G(u)}{w(u)^2}du + \int_c^d \big(D(c,u)D(d,b) - D(u,d)D(a,c)\big)\frac{G(u)}{w(u)^2}du$$

$$+\int_d^b D(c,d)D(u,b)\frac{G(u)}{w(u)^2}du = \int_a^b K(u)G(u)du.$$

Applying the Hölder's inequality the theorem is proved.

Remark 3. *For $w(t) = t$, $t \in [a,b]$, we obtain the result proved by M. Matić and S. Ungar in [6].*

Let us now consider the limit case $d = c =: x$. By the mean value theorem it is reasonable to assume that $\frac{1}{d-c}\int_c^d f(t)dt$ has the value $f(x)$.

Corollary 1. *Let the functions $f, w : [a,b] \to R$ be continuous on $[a,b]$ and differentable on (a,b) and $w(x) \neq 0$ for all $x \in [a,b]$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \le p,q \le \infty$ and $a \le x \le b$, the following inequality holds*

$$\left| f(x)\int_a^b w(t)dt - w(x)\int_a^b f(t)dt \right| \le \|K_x\|_q \|G\|_p, \textit{ where}$$

$$K_x(u) = \begin{cases} -\dfrac{w(x)}{w(u)^2}D(a,u), & ,a \le u \le x, \\ \dfrac{w(x)}{w(u)^2}D(u,b), & ,x < u \le b, \end{cases}$$

$$G(t) = f(t)w'(t) - f'(t)w(t), \quad t \in [a,b].$$

In the next part of the paper we consider the ase when the line segments $[a,b]$ and $[c,d]$ overlap, i.e. $[a,b] \cap [c,d]$ equals $[c,b]$ or $[a,d]$. Now, we will consider the case $a \le c < b \le d$.

For real numbers $\alpha \le \gamma < \lambda \le \beta$ and a real function $\varphi \in L_p[\alpha,\beta]$, $1 \le p \le \infty$, denote by $\|\varphi\|_{p,[\gamma,\lambda]} := \left(\int_\gamma^\lambda |\varphi(t)|^p dt \right)^{1/p}$ the L_p-norm of the restriction of φ to the subinterval $[\gamma,\lambda] \subseteq [\alpha,\beta]$.

Theorem 7. *Let $a \le c < b \le d$ and let the functions $f, w : [a,d] \to R$ be continuous on $[a,d]$ and differentable on (a,d) and $w(x) \ne 0$ for all $x \in [a,d]$. Then for $\frac{1}{p} + \frac{1}{q} = 1$, with $1 \le p, q \le \infty$, the following inequality holds*

(10) $$\left| \int_c^d f(x)dx \int_a^b w(t)dt - \int_c^d w(x)dx \int_a^b f(t)dt \right| \le \|L\|_{p,[a,d]} \|G\|_{q,[a,d]},$$ *where*

$$L(u) = \begin{cases} \frac{1}{w(u)^2} D(d,c)D(a,u), & a \le u \le c, \\ \frac{1}{w(u)^2} \big(D(c,u)D(d,b) - D(u,d)D(a,c) \big), & c \le u \le b, \\ -\frac{1}{w(u)^2} D(u,d)D(a,b), & b \le u \le d, \end{cases}$$

$D(y,z) = \int_y^z w(t)dt, \quad y, z \in [a,d]$, *and* $G(t) = f(t)w'(t) - f'(t)w(t), \ t \in [a,d]$.

Proof. Applying Lemma 1 for the function f and integrating on t over $[a,b]$, and change the order of integration to obtain

$$f(x)\int_a^b w(t)dt - w(x)\int_a^b f(t)dt = -w(x)\int_a^x \frac{D(a,u)}{w(u)^2} G(u)du + w(x)\int_x^b \frac{D(u,b)}{w(u)^2} G(u)du .$$

Integrating this identity on x over $[c,d]$, and changing the order of integration, gives

$$\int_c^d f(x)dx \int_a^b w(t)dt - \int_c^d w(x)dx \int_a^b f(t)dt =$$

$$\int_a^c D(d,c)D(a,u)\frac{G(u)}{w(u)^2} du + \int_c^b \big(D(c,u)D(d,b) - D(u,d)D(a,c) \big) \frac{G(u)}{w(u)^2} du$$

$$-\int_b^d D(u,d)D(a,b)\frac{G(u)}{w(u)^2} du = \int_a^d L(u)G(u)du.$$

Applying the Hölder's inequality the theorem is proved.

Remark 4. *For $w(t) = t$, $t \in [a,b]$ we obtained the M. Matić and S. Ungar's result (see [6]).*

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