*ORIGINAL PAPER*

# TIMELIKE BIHARMONIC S-CURVES ACCORDING TO SABBAN FRAME IN $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$

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***Abstract.*** *In this paper, we study timelike biharmonic curves according to Sabban frame in the* $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$. *We characterize the timelike biharmonic curves in terms of their geodesic curvature. Finally, we find out their explicit parametric equations according to Sabban Frame.*

***Keywords:*** *Biharmonic curve, Heisenberg group, Geodesic curvature.*
***Mathematics Subject Classifications:*** *53C41, 53A10.*

## 1. INTRODUCTION

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} |\mathrm{T}(\phi)|^2 dv_h,$$

where $\mathrm{T}(\phi) := \mathrm{tr}\nabla^{\phi} d\phi$ is the tension field of $\phi$.

The Euler--Lagrange equation of the bienergy is given by $\mathrm{T}_2(\phi) = 0$. Here the section $\mathrm{T}_2(\phi)$ is defined by

$$\mathrm{T}_2(\phi) = -\Delta_\phi \mathrm{T}(\phi) + \mathrm{tr}R(\mathrm{T}(\phi), d\phi)d\phi, \tag{1.1}$$

and called the bitension field of $\phi$. Non-harmonic biharmonic maps are called proper biharmonic maps.

This study is organised as follows: Firstly, we study timelike biharmonic curves accordig to Sabban frame in the Heisenberg group $\mathrm{Heis}^3$. Secondly, we characterize the timelike biharmonic curves in terms of their geodesic curvature and we prove that all of biharmonic curves are helices in the Heisenberg group $\mathrm{Heis}^3$. Finally, we find out their explicit parametric equations according to Sabban Frame.

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## 2. THE LORENTZIAN HEISENBERG GROUP H

Heisenberg group $\mathrm{H}$ can be seen as the space $\mathrm{R}^3$ endowed with the following multiplication:

$$(\bar{x},\bar{y},\bar{z})(x,y,z)=(\bar{x}+x,\bar{y}+y,\bar{z}+z-\frac{1}{2}\bar{x}y+\frac{1}{2}x\bar{y})$$

$\mathrm{Heis}^3$ is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The identity of the group is $(0,0,0)$ and the inverse of $(x,y,z)$ is given by $(-x,-y,-z)$. The left-invariant Lorentz metric on $\mathrm{H}$ is

$$g=-dx^2+dy^2+(xdy+dz)^2.$$

The following set of left-invariant vector fields forms an orthonormal basis for the corresponding Lie algebra:

$$\left\{\mathbf{e}_1=\frac{\partial}{\partial z},\mathbf{e}_2=\frac{\partial}{\partial y}-x\frac{\partial}{\partial z},\mathbf{e}_3=\frac{\partial}{\partial x}\right\}. \tag{2.1}$$

The characterising properties of this algebra are the following commutation relations:

$$g(\mathbf{e}_1,\mathbf{e}_1)=g(\mathbf{e}_2,\mathbf{e}_2)=1,\ g(\mathbf{e}_3,\mathbf{e}_3)=-1.$$

**Proposition 2.1**. For the covariant derivatives of the Levi-Civita connection of the left-invariant metric $g$, defined above the following is true:

$$\nabla=\frac{1}{2}\begin{pmatrix}0 & \mathbf{e}_3 & \mathbf{e}_2\\ \mathbf{e}_3 & 0 & \mathbf{e}_1\\ \mathbf{e}_2 & -\mathbf{e}_1 & 0\end{pmatrix}, \tag{2.2}$$

where the $(i,j)$-element in the table above equals $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ for our basis $\{\mathbf{e}_k, k=1,2,3\}$.

The unit pseudo-Heisenberg sphere (Lorentzian Heisenberg sphere) is defined by

$$\left(\mathrm{S}_1^2\right)_{\mathrm{H}}=\{\boldsymbol{\beta}\in\mathrm{H}:g(\boldsymbol{\beta},\boldsymbol{\beta})=1\}.$$

We adopt the following notation and sign convention for Riemannian curvature operator:

$$R(X,Y)Z=\nabla_X\nabla_Y Z-\nabla_Y\nabla_X Z-\nabla_{X,Y]}Z.$$

The Riemannian curvature tensor is given by

$$R(X,Y,Z,W)=-g(R(X,Y)Z,W).$$

Moreover we put

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \ R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l),$$

where the indices $i, j, k$ and $l$ take the values 1,2 and 3.

$$R_{121} = \frac{1}{4}\mathbf{e}_2, R_{131} = \frac{1}{4}\mathbf{e}_3, \ R_{232} = -\frac{3}{4}\mathbf{e}_3$$

and

$$R_{1212} = -\frac{1}{4}, R_{1313} = \frac{1}{4}, R_{2323} = -\frac{3}{4}. \quad (2.3)$$

## 3. TIMELIKE BIHARMONIC S-CURVES ACCORDING TO SABBAN FRAME IN THE $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$

Let $\gamma : I \to \mathbb{H}$ be a timelike curve in the Lorentzian Heisenberg group $\mathbb{H}$ parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Lorentzian Heisenberg group $\mathbb{H}$ along $\gamma$ defined as follows:

$\mathbf{T}$ is the unit vector field $\gamma'$ tangent to $\gamma$, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to $\gamma$), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\begin{aligned} \nabla_{\mathbf{T}}\mathbf{T} &= \kappa\mathbf{N}, \\ \nabla_{\mathbf{T}}\mathbf{N} &= \kappa\mathbf{T} + \tau\mathbf{B}, \\ \nabla_{\mathbf{T}}\mathbf{B} &= -\tau\mathbf{N}, \end{aligned} \quad (3.1)$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ is its torsion,

$$\begin{gathered} g(\mathbf{T}, \mathbf{T}) = -1, g(\mathbf{N}, \mathbf{N}) = 1, g(\mathbf{B}, \mathbf{B}) = 1, \\ g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0. \end{gathered}$$

Now we give a new frame different from Frenet frame. Let $\alpha : I \to \left(\mathbb{S}_1^2\right)_{\mathbb{H}}$ be unit speed spherical timelike curve. We denote $\sigma$ as the arc-length parameter of $\alpha$ . Let us denote $\mathbf{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of $\alpha$. We now set a vector $\mathbf{s}(\sigma) = \alpha(\sigma) \times \mathbf{t}(\sigma)$ along $\alpha$. This frame is called the Sabban frame of $\alpha$ on $\left(\mathbb{S}_1^2\right)_{\mathbb{H}}$. Then we have the following spherical Frenet-Serret formulae of $\alpha$ :

$$\begin{aligned} \alpha' &= \mathbf{t}, \\ \mathbf{t}' &= \alpha + \kappa_g\mathbf{s}, \\ \mathbf{s}' &= \kappa_g\mathbf{t}, \end{aligned} \quad (3.2)$$

where $\kappa_g$ is the geodesic curvature of the timelike curve $\alpha$ on the $\left(\mathrm{S}_1^2\right)_{\mathrm{H}}$ and

$$g(\mathbf{t},\mathbf{t})=-1, g(\alpha,\alpha)=1, g(\mathbf{s},\mathbf{s})=1,$$
$$g(\mathbf{t},\alpha)=g(\mathbf{t},\mathbf{s})=g(\alpha,\mathbf{s})=0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, we can write

$$\begin{aligned}\alpha &= \alpha_1\mathbf{e}_1+\alpha_2\mathbf{e}_2+\alpha_3\mathbf{e}_3,\\ \mathbf{t} &= t_1\mathbf{e}_1+t_2\mathbf{e}_2+t_3\mathbf{e}_3,\\ \mathbf{s} &= s_1\mathbf{e}_1+s_2\mathbf{e}_2+s_3\mathbf{e}_3.\end{aligned} \tag{3.3}$$

To separate a biharmonic curve according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $\mathrm{S}$-curve.

**Theorem 3.1.**
is a timelike biharmonic S-curve if and only if $\kappa_g$=constant $\neq 0$,

$$\begin{aligned}1+\kappa_g^2 &= [-\frac{1}{4}+\frac{1}{2}s_1^2]+\kappa_g[\alpha_1 s_1],\\ \kappa_g^3 &= \alpha_3 s_3-\kappa_g[\frac{1}{4}-\frac{1}{2}\alpha_1^2].\end{aligned} \tag{3.4}$$

*Proof:* Using (2.1) and Sabban formulas (3.2), we have (3.4).

**Corollary 3.2.** All of timelike biharmonic S-curves in $\left(\mathrm{S}_1^2\right)_{\mathrm{H}}$ are helices.

**Theorem 3.3.** Let $\alpha:\mathrm{I}\to\left(\mathrm{S}_1^2\right)_{\mathrm{H}}$ be a unit speed non-geodesic timelike biharmonic S-curve. Then, the parametric equations of $\alpha$ are

$$\begin{aligned}x^{\mathrm{S}}(\sigma) &= \frac{\cosh\mathrm{A}}{\mathrm{B}_0}\sinh[\mathrm{B}_0\sigma+\mathrm{B}_1]+\mathrm{B}_2,\\ y^{\mathrm{S}}(\sigma) &= \frac{\cosh\mathrm{A}}{\mathrm{B}_0}\cosh[\mathrm{B}_0\sigma+\mathrm{B}_1]+\mathrm{B}_3,\\ z^{\mathrm{S}}(\sigma) &= \sinh\mathrm{A}\sigma+\frac{\cosh^2\mathrm{A}}{2\mathrm{B}_0^2}[\mathrm{B}_0\sigma+\mathrm{B}_1]-\frac{\cosh^2\mathrm{A}}{4\mathrm{B}_0^2}\sinh 2[\mathrm{B}_0\sigma+\mathrm{B}_1]\\ &\quad -\frac{\mathrm{B}_2}{\mathrm{B}_0}\cosh\mathrm{A}\cosh[\mathrm{B}_0\sigma+\mathrm{B}_1]+\mathrm{B}_4,\end{aligned} \tag{3.5}$$

where $\mathrm{B}_1,\mathrm{B}_2,\mathrm{B}_3,\mathrm{B}_4$ are constants of integration and

$$\mathrm{B}_0=\frac{\sqrt{1+\kappa_g^2}}{\cosh\mathrm{A}}-\sinh\mathrm{A}.$$

*Proof:* Since $\alpha$ is timelike biharmonic, $\alpha$ is a $\mathsf{S}-$helix. So, without loss of generality, we take the axis of $\alpha$ is parallel to the vector $\mathbf{e}_1$. Then,

$$g(\mathbf{t},\mathbf{e}_1) = t_1 = \sinh \mathsf{A}, \tag{3.6}$$

where $\mathsf{A}$ is constant angle.

So, substituting the components $t_1$, $t_2$ and $t_3$ in the equation (3.3), we have the following equation

$$\mathbf{t} = \sinh \mathsf{A}\mathbf{e}_1 + \cosh \mathsf{A} \sinh \Pi(\sigma)\mathbf{e}_2 + \cosh \mathsf{A} \cosh \Pi(\sigma)\mathbf{e}_3. \tag{3.7}$$

Using the formula of the Sabban, we write a relation:

$$\nabla_{\mathbf{t}}\mathbf{t} = t_1^{'}\mathbf{e}_1 + (t_2^{'} + t_1 t_3)\mathbf{e}_2 + (t_3^{'} + t_1 t_2)\mathbf{e}_3. \tag{3.8}$$

From above equation, we have

$$\Pi(\sigma) = (\frac{\sqrt{1+\kappa_g^2}}{\cosh \mathsf{A}} - \sinh \mathsf{A})\sigma + \mathsf{B}_1, \tag{3.9}$$

where $\mathsf{B}_1$ is a constant of integration.

Thus (3.7) and (3.9), imply

$$\begin{aligned} \mathbf{t} &= \sinh \mathsf{A}\mathbf{e}_1 + \cosh \mathsf{A} \sinh[(\frac{\sqrt{1+\kappa_g^2}}{\cosh \mathsf{A}} - \sinh \mathsf{A})\sigma + \mathsf{B}_1]\mathbf{e}_2 \\ &+ \cosh \mathsf{A} \cosh[(\frac{\sqrt{1+\kappa_g^2}}{\cosh \mathsf{A}} - \sinh \mathsf{A})\sigma + \mathsf{B}_1]\mathbf{e}_3. \end{aligned} \tag{3.10}$$

Using (2.1) in (3.10), we obtain

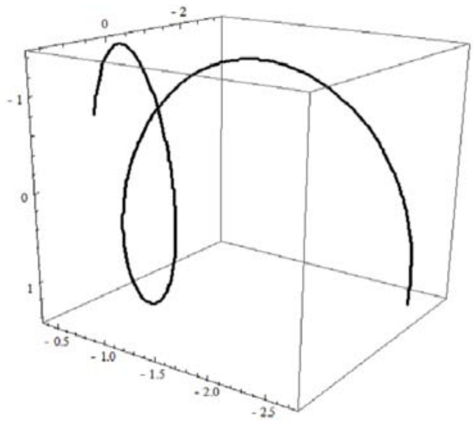
$$\begin{aligned} \mathbf{t} = (&\cosh \mathsf{A} \cosh[\mathsf{B}_0\sigma + \mathsf{B}_1], \cosh \mathsf{A} \sinh[\mathsf{B}_0\sigma + \mathsf{B}_1], \\ &\sinh \mathsf{A} + \cosh \mathsf{A}(\frac{\cosh \mathsf{A}}{\mathsf{B}_0} \sinh[\mathsf{B}_0\sigma + \mathsf{B}_1] + \mathsf{B}_2) \sinh[\mathsf{B}_0\sigma + \mathsf{B}_1]), \end{aligned}$$

where $\mathsf{B}_1, \mathsf{B}_2$ are constants of integration and

$$\mathsf{B}_0 = \frac{\sqrt{1+\kappa_g^2}}{\cosh \mathsf{A}} - \sinh \mathsf{A}.$$

Integrating both sides, we have (3.5). This proves our assertion. Thus, the proof of theorem is completed.

We can use Mathematica in above theorem, yields

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