ORIGINAL PAPER

ON PARTIAL QUASI-BILINEAR GENERATING FUNCTIONS INVOLVING MODIFIED LAGUERRE POLYNOMIALS

PRASANTA KUMAR MAITI [1], ASIT KUMAR CHONGDAR [2]

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Abstract: *In this note we have shown the existence of more general generating relation from the existence of a partial quasi-bilinear generating relation by using group theoretic method. Some particular cases of interest are also pointed out.*

Key words: *Modified Laguerre Polynomial, Generating Function.*

AMS-1991 Subject Classification Code: *33 A75.*

1. INTRODUCTION

In [1], partial quasi-bilateral (quasi-bilinear) generating function is defined as follows:

$$G(x,z,w) = \sum_{n=0}^{\infty} a_n \; w^n \; P_{m+n}^{(\alpha)}(x) \; Q_r^{(m+n)}(z), \tag{1}$$

where the co-efficients a_n are quite arbitrary and $P_{m+n}^{(\alpha)}(x)$,$Q_r^{(m+n)}(z)$ are two particular special functions of orders m+n, r and of parameters α and m+n respectively. If, in particular, $Q_r^{(m+n)}(z) \equiv P_r^{(m+n)}(z)$, then it is quasi-bilinear.

In the present paper, we have shown the existence of a more general generating function involving modified Laguerre polynomials from the existence of a partial quasi-bilinear generating function involving the polynomial under consideration from the group theoretic view point.

In [2], Sharma and Chongdar obtained the following theorem on bilateral generating functions involving the modified Laguerre polynomials as introduced by Singh and Bala [3]:

Theorem 1. If there exists a linear(unilateral) generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n \; L_{a,b,m,n}(x) \; w^n \tag{2}$$

then

$$(1-wb)^{-m} \; exp\left(\frac{-wax}{1-wb}\right) G\left(\frac{x}{1-wb}, \frac{wz}{1-wb}\right) = \sum_{n=0}^{\infty} \sigma_n(z) \; L_{a,b,m,n}(x) \; w^n, \tag{3}$$

where

[1] Bangabasi Evening College, Department of Physics, Kolkata, 700009, India.
E-mail: maitiprasantakumar@gmail.com.
[2] Bengal Engineering and Science University, Shibpur, 711103, India. E-mail: chongdarmath@yahoo.co.in.

$$\sigma_n(z) = \sum_{k=0}^{n} a_k \binom{n}{k} z^k. \tag{4}$$

In [4], the following theorem in connection with the extension of the above theorem has been obtained by the authors of [2].

Theorem 2. If there exists an unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n \; L_{a,b,m,n+k}(x)\, w^n, \tag{5}$$

where k is a non-negative integer, then we have

$$(1-wb)^{-m-k} \; exp\left(\frac{-wax}{1-wb}\right) G\left(\frac{x}{1-wb}, \frac{wz}{1-wb}\right)$$
$$= \sum_{n=0}^{\infty} \sigma_n(z) \; L_{a,b,m,n+k}(x) \; w^n, \tag{6}$$

where

$$\sigma_n(z) = \sum_{p=0}^{n} a_p \binom{n+k}{p+k} z^p. \tag{7}$$

The importance of the above theorems lies in the fact that whenever one knows a generating relation of the type (2) or (5) the corresponding bilateral or extension of the bilateral generating relation is at once be written down from (3) or (6). Thus one can get a large number of bilateral or extension of bilateral generating relations from (3) or (6) by attributing different suitable values to a_n in (2) or (5).

In this article, the above theorem has been further extended from the concept of partial quasi-bilinear generating function as defined in [1] by using one parameter group of continuous transformations. The main result of the paper is stated in the form of the following theorem:

Theorem 3. If

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n \; L_{a,b,m,n+r}(x) \; L_{a,b,n+r,k}(u) \; w^n \tag{8}$$

then

$$exp\left[-wt - \frac{waxy}{1-wby}\right] (1-wby)^{-m-r} \; G\left(\frac{x}{1-wby}, u + \frac{bwt}{a}, \frac{wytv}{1-wby}\right)$$
$$= \sum_{n,p,q=0}^{\infty} a_n \frac{w^{p+q+n}}{p!q!} (n+r+1)_p \; L_{a,b,m,n+r+p}(x)\, y^{n+p}$$
$$\times (-1)^q \; L_{a,b,n+r+q,k}(u)\, t^{n+q}\, v^n. \tag{9}$$

2. PROOF OF THE THEOREM 3

For the modified Laguerre polynomials we consider the following operators [4, 5]:

$$R_1 = bxy\frac{\partial}{\partial x} + by^2\frac{\partial}{\partial y} + y(b(m+r) - ax), \tag{10}$$

$$R_2 = \frac{bt}{a}\frac{\partial}{\partial u} - t \tag{11}$$

such that

$$R_1(L_{a,b,m,n+r}(x)\ y^n) = (n+r+1)\ L_{a,b,m,n+r+1}(x)\ y^{n+1}, \tag{12}$$

$$R_2(L_{a,b,n+r,k}(u)\ t^n) = -L_{a,b,n+r+1,k}(u)\ t^{n+1} \tag{13}$$

and

$$e^{wR_1} f(x,y) = (1-wby)^{-m-r}\ exp\left(\frac{-waxy}{1-wby}\right)\ f\left(\frac{x}{1-wby}, \frac{y}{1-wby}\right), \tag{14}$$

$$e^{wR_2} f(u,t) = e^{-wt}\ \ f\left(u + \frac{b}{a}wt, t\right). \tag{15}$$

Now we consider the following formula

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\ \ w^n\ \ L_{a,b,m,n+r}(x)\ L_{a,b,n+r,k}(u). \tag{16}$$

Replacing w by wytv, we get

$$G(x,u,wytv) = \sum_{n=0}^{\infty} a_n\ \ (wv)^n\ \ (L_{a,b,m,n+r}(x)\ \ y^n)\ \ (L_{a,b,n+r,k}(u)\ \ t^n). \tag{17}$$

Operating $e^{wR_1}e^{wR_2}$ on both sides of (17), we get

$$e^{wR_1}e^{wR_2}(G(x,u,wytv)) = e^{wR_1}e^{wR_2}\sum_{n=0}^{\infty} a_n\ \ (wv)^n\ \ (L_{a,b,m,n+r}(x)\ \ y^n)$$
$$\times\ (L_{a,b,n+r,k}(u)\ \ t^n). \tag{18}$$

Now L.H.S. of (18) is

$$e^{wR_1}e^{wR_2}(G(x,u,wytv))$$
$$= e^{-wt}\ (1-wby)^{-m-r}\ exp\left(\frac{-waxy}{1-wby}\right) G\left(\frac{x}{1-wby}, u + \frac{bwt}{a}, \frac{wytv}{1-wby}\right). \tag{19}$$

R.H.S. of (18) is

$$
\begin{aligned}
& e^{wR_1} e^{wR_2} \sum_{n=0}^{\infty} a_n \, (wv)^n \, (L_{a,b,m,n+r}(x) \, y^n) \, (L_{a,b,n+r,k}(u) \, t^n) \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} a_n \, (wv)^n \, \frac{w^p R_1^p}{p!} \frac{w^q R_2^q}{q!} (L_{a,b,m,n+r}(x) \, y^n) \, (L_{a,b,n+r,k}(u) \, t^n) \\
&= \sum_{n,p,q=0}^{\infty} a_n \, \frac{w^{n+p+q}}{p! q!} R_1^p (L_{a,b,m,n+r}(x) \, y^n) \; R_2^q (L_{a,b,n+r,k}(u) \, t^n) \, v^n \\
&= \sum_{n,p,q=0}^{\infty} a_n \, \frac{w^{n+p+q}}{p! q!} (n+r+1)_p \, L_{a,b,m,n+r+p}(x) \, y^{n+p} \, (-1)^q \, L_{a,b,n+r+q,k}(u) \, t^{n+q} \, v^n. \quad (20)
\end{aligned}
$$

Equating (19) and (20), we get

$$
\begin{aligned}
& exp\left[-wt - \frac{waxy}{1-wby} \right] (1-wby)^{-m-r} \; G\left(\frac{x}{1-wby}, u + \frac{bwt}{a}, \frac{wytv}{1-wby} \right) \\
&= \sum_{n,p,q=0}^{\infty} a_n \, \frac{w^{p+q+n}}{p! q!} \, (n+r+1)_p \; L_{a,b,m,n+r+p}(x) \; y^{n+p} \\
& \times (-1)^q \; L_{a,b,n+r+q,k}(u) \; t^{n+q} \; v^n. \qquad (21)
\end{aligned}
$$

This completes the proof of the theorem.

Corollary 1. Putting $k = 0$ and $y = t = 1$ in (21), we get

$$
\begin{aligned}
& exp\left[-w - \frac{wax}{1-wb} \right] (1-wb)^{-m-r} \; G\left(\frac{x}{1-wb}, \frac{wv}{1-wb} \right) \\
&= \sum_{n,p,q=0}^{\infty} a_n \, \frac{w^{p+q+n}}{p! q!} \, (n+r+1)_p \; (-1)^q \; L_{a,b,m,n+r+p}(x) \; v^n \\
&= \sum_{q=0}^{\infty} \frac{(-w)^q}{q!} \sum_{n,p=0}^{\infty} a_n \, \frac{w^{n+p}}{p!} \, (n+r+1)_p \; L_{a,b,m,n+r+p}(x) \; v^n \\
&= e^{-w} \sum_{n,p=0}^{\infty} a_n \, \frac{w^{n+p}}{p!} \, (n+r+1)_p \; L_{a,b,m,n+r+p}(x) \; v^n.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
& exp\left[- \frac{wax}{1-wb} \right] (1-wb)^{-r-m} \; G\left(\frac{x}{1-wb}, \frac{wv}{1-wb} \right) \\
&= \sum_{n,p=0}^{\infty} a_n \, \frac{w^{n+p}}{p!} \, (n+r+1)_p \; L_{a,b,m,n+r+p}(x) \; v^n \\
&= \sum_{n=0}^{\infty} \sum_{p=0}^{n} a_{n-p} \, \frac{w^n}{p!} \, (n-p+r+1)_p \; L_{a,b,m,n+r}(x) \; v^{n-p} \\
&= \sum_{n=0}^{\infty} w^n \; L_{a,b,m,n+r}(x) \; \sum_{p=0}^{n} a_{n-p} \, \frac{(n-p+r+1)_p}{p!} \; v^{n-p}.
\end{aligned}
$$

Hence

$$exp\left[-\frac{wax}{1-wb}\right]\ (1-wb)^{-r-m}\ \ G\left(\frac{x}{1-wb},\frac{wv}{1-wb}\right)$$
$$=\sum_{n=0}^{\infty} w^n\ \ L_{a,b,m,n+r}(x)\ \ \sigma_n(v)\,, \tag{22}$$

where

$$\sigma_n(v)=\sum_{p=0}^{n} a_p \binom{n+r}{p+r} v^p\,, \tag{23}$$

which is Theorem 2.

Special case 1.

On specialising the parameters as $a=b=1$ and $m=1+\alpha$, in (22) we get the following result:

Result 1. If

$$G(x,w)=\sum_{n=0}^{\infty} a_n\ \ L_{n+r}^{(\alpha)}(x)\ \ w^n$$

then

$$exp\left[-\frac{wx}{1-w}\right]\ (1-w)^{-1-r-\alpha}\ \ G\left(\frac{x}{1-w},\frac{wv}{1-w}\right)$$

$$=\sum_{n=0}^{\infty} w^n\ \ L_{n+r}^{(\alpha)}(x)\ \ \sigma_n(v)\,, \tag{24}$$

where

$$\sigma_n(v)=\sum_{p=0}^{n} a_p \binom{n+r}{p+r} v^p.$$

Which is found derived in [6].

Corollary 2. Putting $k=0,\ r=0$ and $y=t=1$ in (21), we get

$$exp\left[-\frac{wax}{1-wb}\right]\ (1-wb)^{-m}\ \ G\left(\frac{x}{1-wb},\frac{wv}{1-wb}\right)$$
$$=\sum_{n=0}^{\infty} w^n\ \ L_{a,b,m,n}(x)\ \ \sigma_n(v) \tag{25}$$

where

$$\sigma_n(v)=\sum_{p=0}^{n} a_p \binom{n}{p} v^p.$$

Which is Theorem 1.

Special case 2.

On specialising the parameters as $a = b = 1$ and $m = 1+\alpha$, in (25) we get the following result:

Result 2.

If $G(x,w) = \sum_{n=0}^{\infty} a_n \ L_n^{(\alpha)}(x) \ w^n$

then

$$exp\left[-\frac{wx}{1-w}\right] (1-w)^{-1-\alpha} \ G\left(\frac{x}{1-w}, \frac{wv}{1-w}\right)$$
$$= \sum_{n=0}^{\infty} w^n \ L_n^{(\alpha)}(x) \ \sigma_n(v), \qquad (26)$$

where

$$\sigma_n(v) = \sum_{p=0}^{n} a_p \binom{n}{p} v^p.$$

Which is found derived in [7].

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