*ORIGINAL PAPER*

# INEXTENSIBLE FLOWS OF b-TANGENT DEVELOPABLE SURFACES OF BIHARMONIC NEW TYPE b-SLANT HELICES ACCORDING TO BISHOP FRAME IN THE SOL SPACE $\mathbf{Sol}^3$

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*Manuscript received: 11.04.2012; Accepted paper: 05.05.2012;*

*Published online: 15.06.2012.*

***Abstract.*** *In this paper, we study inextensible flows of* $\mathsf{b}-$*tangent developable surfaces of biharmonic new type* $\mathsf{b}-$*slant helix in the* $\mathbf{Sol}^3$. *We obtain partial differential equation for* $-$*tangent developable surfaces of biharmonic new type* $\mathsf{b}-$*slant helix in terms of their Bishop curvatures. Finally, we find explicit parametric equations one-parameter family of the* $\mathsf{b}-$*tangent developable surface of a unit speed non-geodesic biharmonic new type* $\mathsf{b}-$*slant helix in the* $\mathbf{Sol}^3$.

***Keywords:*** *New type* $\mathsf{b}-$*slant helix, Sol Space, Curvature, Torsion.*

***Mathematics Subject Classifications:*** *53A04, 53A10.*

## 1. INTRODUCTION

Recently, the term of flows of surfaces becomes popular. The development of 3D acquisition technologies and computational power, conformal geometry plays more and more important roles in engineering fields. For example, conformal geometry has been broadly applied in computer graphics, computer vision, geometric modeling and medical imaging. The theoretic foundation for computational conformal geometry is developing rapidly and many practical algorithms converting classical theories in conformal geometry have been invented.

In this paper, we study inextensible flows of $\mathsf{b}-$tangent developable surfaces of biharmonic new type $\mathsf{b}-$slant helix in the $\mathbf{Sol}^3$. We obtain partial differential equation for $-$tangent developable surfaces of biharmonic new type $\mathsf{b}-$slant helix in terms of their Bishop curvatures. Finally, we find explicit parametric equations one-parameter family of the $\mathsf{b}-$tangent developable surface of a unit speed non-geodesic biharmonic new type $\mathsf{b}-$slant helix in the $\mathbf{Sol}^3$.

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## 2. RIEMANNIAN STRUCTURE OF SOL SPACE $\mathbf{Sol}^3$

Sol space, one of Thurston's eight 3-dimensional geometries, can be viewed as $\mathbb{R}^3$ provided with Riemannian metric

$$g_{\mathbf{Sol}^3} = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2,$$

where $(x, y, z)$ are the standard coordinates in $\mathbb{R}^3$.

Note that the Sol metric can also be written as:

$$g_{\mathbf{Sol}^3} = \sum_{i=1}^{3}\boldsymbol{\omega}^i \otimes \boldsymbol{\omega}^i,$$

where

$$\boldsymbol{\omega}^1 = e^z dx,\ \boldsymbol{\omega}^2 = e^{-z}dy,\ \boldsymbol{\omega}^3 = dz,$$

and the orthonormal basis dual to the 1-forms is

$$\mathbf{e}_1 = e^{-z}\frac{\partial}{\partial x},\ \mathbf{e}_2 = e^{z}\frac{\partial}{\partial y},\ \mathbf{e}_3 = \frac{\partial}{\partial z}. \tag{2.1}$$

**Proposition 2.1.** *For the covariant derivatives of the Levi-Civita connection of the left-invariant metric* $g_{\mathbf{Sol}^3}$, *defined above the following is true:*

$$\nabla = \begin{pmatrix} -\mathbf{e}_3 & 0 & \mathbf{e}_1 \\ 0 & \mathbf{e}_3 & -\mathbf{e}_2 \\ 0 & 0 & 0 \end{pmatrix}, \tag{2.2}$$

*where the* $(i, j)$*-element in the table above equals* $\nabla_{\mathbf{e}_i}\mathbf{e}_j$ *for our basis*

$$\{\mathbf{e}_k, k = 1,2,3\} = \{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}.$$

Lie brackets can be easily computed as:

$$[\mathbf{e}_1, \mathbf{e}_2] = 0,\ [\mathbf{e}_2, \mathbf{e}_3] = -\mathbf{e}_2,\ [\mathbf{e}_1, \mathbf{e}_3] = \mathbf{e}_1.$$

The isometry group of $\mathbf{Sol}^3$ has dimension 3. The connected component of the identity is generated by the following three families of isometries:

$$(x, y, z) \to (x + c, y, z),$$
$$(x, y, z) \to (x, y + c, z),$$
$$(x, y, z) \to (e^{-c}x, e^{c}y, z + c).$$

## 3. BIHARMONIC NEW TYPE b-SLANT HELICES IN SOL SPACE $\mathbf{Sol}^3$

Assume that $\{\mathbf{t},\mathbf{n},\mathbf{b}\}$ be the Frenet frame field along $\gamma$. Then, the Frenet frame satisfies the following Frenet--Serret equations:

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= \kappa\mathbf{n}, \\ \nabla_{\mathbf{t}}\mathbf{n} &= -\kappa\mathbf{t}+\tau\mathbf{b}, \\ \nabla_{\mathbf{t}}\mathbf{b} &= -\tau\mathbf{n}, \end{aligned} \tag{3.1}$$

where $\kappa$ is the curvature of $\gamma$ and $\tau$ its torsion and

$$\begin{gathered} g_{\mathbf{Sol}^3}(\mathbf{t},\mathbf{t})=1, g_{\mathbf{Sol}^3}(\mathbf{n},\mathbf{n})=1, g_{\mathbf{Sol}^3}(\mathbf{b},\mathbf{b})=1, \\ g_{\mathbf{Sol}^3}(\mathbf{t},\mathbf{n})=g_{\mathbf{Sol}^3}(\mathbf{t},\mathbf{b})=g_{\mathbf{Sol}^3}(\mathbf{n},\mathbf{b})=0. \end{gathered} \tag{3.2}$$

The Bishop frame or parallel transport frame is an alternative approach to defining a moving frame that is well defined even when the curve has vanishing second derivative. The Bishop frame is expressed as

$$\begin{aligned} \nabla_{\mathbf{t}}\mathbf{t} &= k_1\mathbf{m}_1+k_2\mathbf{m}_2, \\ \nabla_{\mathbf{t}}\mathbf{m}_1 &= -k_1\mathbf{t}, \\ \nabla_{\mathbf{t}}\mathbf{m}_2 &= -k_2\mathbf{t}, \end{aligned} \tag{3.3}$$

where

$$\begin{gathered} g_{\mathbf{Sol}^3}(\mathbf{t},\mathbf{t})=1, g_{\mathbf{Sol}^3}(\mathbf{m}_1,\mathbf{m}_1)=1, g_{\mathbf{Sol}^3}(\mathbf{m}_2,\mathbf{m}_2)=1, \\ g_{\mathbf{Sol}^3}(\mathbf{t},\mathbf{m}_1)=g_{\mathbf{Sol}^3}(\mathbf{t},\mathbf{m}_2)=g_{\mathbf{Sol}^3}(\mathbf{m}_1,\mathbf{m}_2)=0. \end{gathered} \tag{3.4}$$

Here, we shall call the set $\{\mathbf{t},\mathbf{m}_1,\mathbf{m}_1\}$ as Bishop trihedra, $k_1$ and $k_2$ as Bishop curvatures and $\delta(s)=\arctan\frac{k_2}{k_1}$, $\tau(s)=\delta'(s)$ and $\kappa(s)=\sqrt{k_1^2+k_2^2}$.

Bishop curvatures are defined by

$$\begin{aligned} k_1 &= \kappa(s)\cos\delta(s), \\ k_2 &= \kappa(s)\sin\delta(s). \end{aligned}$$

The relation matrix may be expressed as

$$\begin{aligned} \mathbf{t} &= \mathbf{t}, \\ \mathbf{n} &= \cos\delta(s)\mathbf{m}_1+\sin\delta(s)\mathbf{m}_2, \\ \mathbf{b} &= -\sin\delta(s)\mathbf{m}_1+\cos\delta(s)\mathbf{m}_2. \end{aligned}$$

On the other hand, using above equation we have

$$\mathbf{t} = \mathbf{t},$$
$$\mathbf{m}_1 = \cos\delta(s)\mathbf{n} - \sin\delta(s)\mathbf{b}$$
$$\mathbf{m}_2 = \sin\delta(s)\mathbf{n} + \cos\delta(s)\mathbf{b}.$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ we can write

$$\mathbf{t} = t^1 e_1 + t^2 e_2 + t^3 e_3,$$
$$\mathbf{m}_1 = m_1^1 \mathbf{e}_1 + m_1^2 \mathbf{e}_2 + m_1^3 \mathbf{e}_3, \quad (3.5)$$
$$\mathbf{m}_2 = m_2^1 \mathbf{e}_1 + m_2^2 \mathbf{e}_2 + m_2^3 \mathbf{e}_3.$$

**Theorem 3.1.** *$\gamma : I \to \mathbf{Sol}^3$ is a biharmonic curve according to Bishop frame if and only if*

$$k_1^2 + k_2^2 = \text{constant} \neq 0,$$
$$k_1^{''} - \left[k_1^2 + k_2^2\right]k_1 = -k_1\left[2m_2^3 - 1\right] - 2k_2 m_1^3 m_2^3, \quad (3.6)$$
$$k_2^{''} - \left[k_1^2 + k_2^2\right]k_2 = 2k_1 m_1^3 m_2^3 - k_2\left[2m_1^3 - 1\right].$$

**Definition 3.2.** *A regular curve $\gamma : I \to \mathbf{Sol}^3$ is called a new type slant helix provided the unit vector $\mathbf{m}_2$ of the curve $\gamma$ has constant angle $\mathrm{M}$ with some fixed unit vector $u$, that is*

$$g_{\mathbf{Sol}^3}(\mathbf{m}_2(s), u) = \cos\mathrm{M} \text{ for all } s \in I. \quad (3.7)$$

The condition is not altered by reparametrization, so without loss of generality we may assume that new type slant helices have unit speed. The second slant helices can be identified by a simple condition on natural curvatures.

To separate a new type slant helix according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for the curve defined above as new type $\mathrm{b}-$ slant helix.

We shall also use the following lemma.

**Theorem 3.3.** *Let $\gamma : I \to \mathbf{Sol}^3$ be a unit speed curve. Then $\gamma$ is a new type $\mathrm{b}-$slant helix if and only if*

$$k_1 = -k_2 \cot\mathrm{M}. \quad (3.8)$$

## 4. INEXTENSIBLE FLOWS OF b-TANGENT DEVELOPABLE SURFACES OF BIHARMONIC NEW TYPE b-SLANT HELICES IN SOL SPACE $\mathbf{Sol}^3$

To separate a tangent developable according to Bishop frame from that of Frenet-Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathrm{b}-$tangent developable.

The purpose of this section is to study $\mathrm{b}-$tangent developable of biharmonic new type $\mathrm{b}-$slant helix in $\mathbf{Sol}^3$.

The $\mathrm{b}-$tangent developable of $\gamma$ is a ruled surface

$$\mathrm{Y}_{new}(s,u)=\gamma(s)+u\gamma'(s). \tag{4.1}$$

**Definition 4.1.** *A surface evolution* $\mathrm{Y}_{new}(s,u,t)$ *and its flow* $\frac{\partial \mathrm{Y}_{new}}{\partial t}$ *are said to be inextensible if its first fundamental form* $\{\mathbf{E},\mathbf{F},\mathbf{G}\}$ *satisfies*

$$\frac{\partial \mathbf{E}}{\partial t}=\frac{\partial \mathbf{F}}{\partial t}=\frac{\partial \mathbf{G}}{\partial t}=0. \tag{4.2}$$

**Definition 4.2.** *We can define the following one-parameter family of developable ruled surface*

$$\mathrm{Y}_{new}=\gamma(s,t)+u\gamma'(s,t). \tag{4.3}$$

Hence, we have the following theorem.

**Theorem 4.3.** *Let* $\mathrm{Y}_{new}$ *be one-parameter family of the* $\mathrm{b}-$*tangent developable of a unit speed non-geodesic biharmonic new type* $\mathrm{b}-$*slant helix. Then* $\frac{\partial \mathrm{Y}_{new}}{\partial t}$ *is inextensible if and only if*

$$\begin{aligned}
&\frac{\partial}{\partial t}(\cos \mathrm{M}(t)\sin[\mathrm{S}_1(t)s+\mathrm{S}_2(t)]+uk_1(t)\cos[\mathrm{S}_1(t)s+\mathrm{S}_2(t)]\\
&+uk_2(t)\sin \mathrm{M}(t)\sin[\mathrm{S}_1(t)s+\mathrm{S}_2(t)])^2+\frac{\partial}{\partial t}(\cos \mathrm{M}(t)\cos[\mathrm{S}_1(t)s+\mathrm{S}_2(t)]\\
&-uk_1(t)\sin[\mathrm{S}_1(t)s+\mathrm{S}_2(t)]+uk_2(t)\sin \mathrm{M}(t)\cos[\mathrm{S}_1(t)s+\mathrm{S}_2(t)])^2\\
&=-\frac{\partial}{\partial t}(-\sin \mathrm{M}(t)+uk_2(t)\cos \mathrm{M}(t))^2,
\end{aligned} \tag{4.4}$$

*Proof:* Assume that $\mathrm{Y}(s,u,t)$ be a one-parameter family of the $\mathrm{b}-$tangent developable of a unit speed non-geodesic biharmonic new type $\mathrm{b}-$slant helix.

From our assumption, we get the following equation

$$\mathbf{m}_2 = \sin \mathrm{M}(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)]\mathbf{e}_1 + \sin \mathrm{M}(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)]\mathbf{e}_2 + \cos \mathrm{M}(t)\mathbf{e}_3, \tag{4.5}$$

where $\mathrm{S}_1, \mathrm{S}_2$ are smooth functions of time.

On the other hand, using Bishop formulas Eq.(3.3) and Eq.(2.1), we have

$$\mathbf{m}_1 = \cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)]\mathbf{e}_1 - \sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)]\mathbf{e}_2. \tag{4.6}$$

Using above equation and Eq.(4.5), we get

$$\mathbf{t} = \cos \mathrm{M}(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)]\mathbf{e}_1 + \cos \mathrm{M}(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)]\mathbf{e}_2 - \sin \mathrm{M}(t)\mathbf{e}_3. \tag{4.7}$$

Furthermore, we have the natural frame $\{\mathrm{Y}_s, \mathrm{Y}_u\}$ given by

$$\begin{aligned}\mathrm{Y}_s = {} & (\cos \mathrm{M}(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)] + uk_1(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)] \\ & + uk_2(t)\sin \mathrm{M}(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)])\mathbf{e}_1 + (\cos \mathrm{M}(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)] \\ & - uk_1(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)] + uk_2(t)\sin \mathrm{M}(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)])\mathbf{e}_2 \\ & + (-\sin \mathrm{M}(t) + uk_2(t)\cos \mathrm{M}(t))\mathbf{e}_3,\end{aligned} \tag{4.8}$$

and

$$\mathrm{Y}_u = \cos \mathrm{M}(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)]\mathbf{e}_1 + \cos \mathrm{M}(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)]\mathbf{e}_2 - \sin \mathrm{M}(t)\mathbf{e}_3.$$

The components of the first fundamental form are

$$\begin{aligned}\frac{\partial \mathbf{E}}{\partial t} = {} & \frac{\partial}{\partial t} g_{\mathbf{Sol}^3}(\mathrm{Y}_s, \mathrm{Y}_s) = \frac{\partial}{\partial t}(\cos \mathrm{M}(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)] + uk_1(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)] \\ & + uk_2(t)\sin \mathrm{M}(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)])^2 + \frac{\partial}{\partial t}(\cos \mathrm{M}(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)] \\ & - uk_1(t)\sin[\mathrm{S}_1(t)s + \mathrm{S}_2(t)] + uk_2(t)\sin \mathrm{M}(t)\cos[\mathrm{S}_1(t)s + \mathrm{S}_2(t)])^2 \\ & + \frac{\partial}{\partial t}(-\sin \mathrm{M}(t) + uk_2(t)\cos \mathrm{M}(t))^2,\end{aligned}$$

$$\frac{\partial \mathbf{F}}{\partial t} = 0, \tag{4.9}$$

$$\frac{\partial \mathbf{G}}{\partial t} = 0.$$

Hence, $\frac{\partial \mathrm{Y}}{\partial t}$ is inextensible if and only if Eq.(4.4) is satisfied. This concludes the proof of theorem.

**Theorem 4.4.** *Let* $\Upsilon_{new}$ *be one-parameter family of the* $\mathrm{b}-$*tangent developable surface of a unit speed non-geodesic biharmonic new type* $\mathrm{b}-$*slant helix. Then, the parametric equations of this family are given by*

$$\mathbf{x}(s,u,t)=-\frac{\cos M(t)e^{\sin M(t)s-S_3(t)}}{S_1^2(t)+\sin^2 M(t)}S_1(t)\cos[S_1(t)s+S_2(t)]$$
$$+\frac{\cos M(t)e^{\sin M(t)s-S_3(t)}}{S_1^2(t)+\sin^2 M(t)}\sin M(t)\sin[S_1(t)s+S_2]$$
$$+u\cos M(t)\sin[S_1(t)s+S_2(t)]e^{\sin M(t)s-S_3(t)}+S_4(t),$$

$$\mathbf{y}(s,u,t)=-\frac{\cos M(t)e^{-\sin M(t)s+S_3(t)}}{S_1^2(t)+\sin^2 M(t)}\sin M(t)\cos[S_1(t)s+S_2(t)] \tag{4.10}$$
$$+\frac{\cos M(t)e^{-\sin M(t)s+S_3(t)}}{S_1^2(t)+\sin^2 M(t)}S_1(t)\sin[S_1(t)s+S_2(t)]$$
$$+u\cos M(t)\cos[S_1(t)s+S_2(t)]e^{-\sin M(t)s+S_3(t)}+S_5(t),$$

$$\mathbf{z}(s,u,t)=-\sin M(t)s-u\sin M(t)+S_3(t),$$

*where* $S_1,S_2,S_3,S_4,S_5$ *are smooth functions of time.*

*Proof:* We assume that $\gamma$ is a unit speed new type $\mathrm{b}-$slant helix.

Substituting Eq.(2.4) to Eq.(4.7), we have

$$\mathbf{t}=(\cos M(t)\sin[S_1(t)s+S_2(t)]e^{-z},\cos M(t)\cos[S_1(t)s+S_2(t)]e^{z},$$
$$-\sin M(t)).$$

Substituting this into the Eq.(4.3), we have Eq.(4.10). Thus, the proof is completed.

We can use Mathematica in above theorem, yields

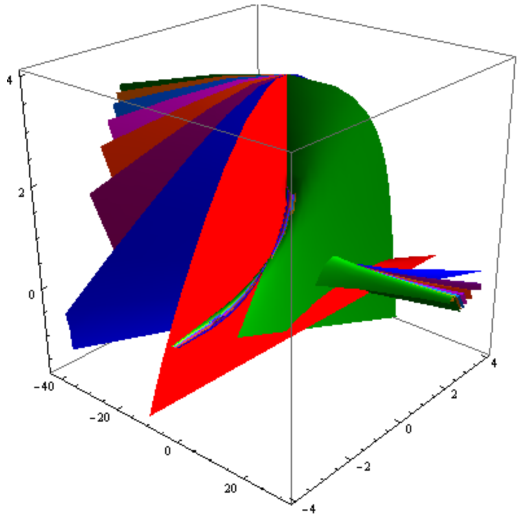


**Fig. 1.**

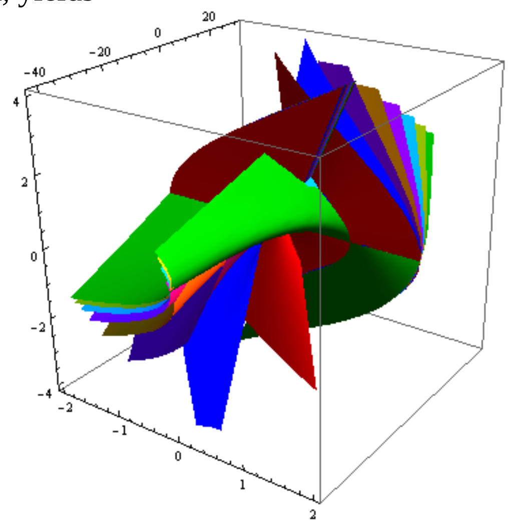


**Fig. 2.**

**Fig. 1,2: The equation (4.10) is illustrated colour Red, Blue, Purple, Orange, Magenta, Cyan, Yellow, Green at the time t=1, t=1.2, t=1.4, t=1.6, t=1.8, t=2, t=2.2, t=2.4, respectively.**

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