ORIGINAL PAPER

ON MIXED TRILATERAL GENERATING FUNCTIONS OF CERTAIN SPECIAL FUNCTION

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Abstract. *In this note, we have obtained a novel result which is stated in the form of a theorem on mixed trilateral generating relations involving modified Laguerre polynomials. As a particular case, we obtain an interesting result, which is worthy of notice.*

Keywords: *Modified Laguerre Polynomial, Generating Function.*

AMS- 1991 subject classification code: *33A75*

1. INTRODUCTION

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The modified Laguerre polynomial $f_n^{\beta}(x)$ is defined by [1]:

$$f_n^{\beta}(x) = \frac{(\beta)_n}{n!}\ {}_1F_1\left[\begin{matrix} -n\,; \\ & x \\ 1-\beta-n\,; \end{matrix}\right]. \tag{1.1}$$

The object of the present paper is to obtain a novel result on bilateral generating functions involving modified Laguerre polynomials, $f_n^{\beta+n}(x)$ where $f_n^{\beta}(x)$ is defined by (1.1), from the Lie group view point. For previous works on $f_n^{\beta}(x)$ by group-theoretic method ,one can see the works [3-7].

The main result of this note is stated in the form of the following theorem .

Theorem 1. If there exists a bilateral generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n\ f_n^{\beta+n}(x)\ g_n(u)\ w^n \quad , \tag{1.2}$$

where $g_n(u)$ is an arbitrary polynomials of degree n, then

$$(1+tz)^{\beta-1}\ \exp\left\{\frac{txz}{1+tz}\right\} G\left(\frac{x}{1+tz}, u, (1+tz)t\right) = \sum_{n=0}^{\infty} t^n \sigma_n(x,u,z) \quad , \tag{1.3}$$

where

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$$\sigma_n(x,u,z) = \sum_{p=0}^{n} a_p \binom{n}{p} f_n^{\beta-n+2p}(x)\, g_p(u)\, z^{n-p} \quad . \tag{1.4}$$

The importance of the above theorem lies in the fact that whenever one knows a bilateral generating relation of the form (1.2), the corresponding mixed trilateral generating relation can at once be written down from the relation (1.3). Thus one can get a large number of mixed trilateral generating relations by attributing different suitable values to a_n in (1.2).

2. DERIVATION OF THE GENERATING FUNCTION

For the modified Laguerre polynomials, we consider the following partial differential operator [2] :

$$R = xyz^{-2}\frac{\partial}{\partial x} - y^2 z^{-2}\frac{\partial}{\partial y} - yz^{-1}\frac{\partial}{\partial z} + (1-x)yz^{-2} \tag{2.1}$$

such that

$$R\left(f_n^{\beta+n}(x)\, y^n z^{\beta} \right) = -(n-1) f_n^{\beta+n-1}(x) y^{n+1} z^{\beta-2} \ , \tag{2.2}$$

and

$$e^{wR}\ f(x,y,z) =$$

$$= \left(1-\frac{wy}{z^2}\right)^{-1} \exp\left\{\frac{-(wxy)/z^2}{1-(wy)/z^2}\right\} f\left(\frac{x}{\left(1-(wy)/z^2\right)}, \left(1-(wy)/z^2\right)y\ , \left(1-(wy)/z^2\right)z\right). \tag{2.3}$$

Now using (2.3), we get

$$e^{wR}\left(f_n^{\beta+n}(x)\, y^n z^{\beta} \right) = \left(1-\frac{wy}{z^2}\right)^{-1+n+\beta} \exp\left\{\frac{-(wxy)/z^2}{1-(wy)/z^2}\right\} f_n^{\beta+n}\left(\frac{x}{1-(wy)/z^2}\right) y^n z^{\beta}. \tag{2.4}$$

Again by using (2.2), we have

$$\begin{aligned} e^{wR}\left(f_n^{\beta+n}(x)\, y^n z^{\beta} \right) &= \sum_{p=0}^{\infty} \frac{w^p}{p!}\ R^p\left(f_n^{\beta+n}(x)\, y^n z^{\beta} \right) \\ &= \sum_{p=0}^{\infty} \frac{w^p}{p!} (-1)^p\ (n+1)_p\ f_{n+p}^{\beta+n-p}(x)\ y^{n+p}\ z^{\beta-2p} \\ &= y^n z^{\beta} \sum_{p=0}^{\infty} \left(\frac{-yw}{z^2}\right)^p \frac{(n+1)_p}{p!} f_{n+p}^{\beta+n-p}(x). \end{aligned} \tag{2.5}$$

Equating (2.4) and (2.5) then putting $-\dfrac{wy}{z^2} = t$, we obtain

$$(1+t)^{n+\beta-1}\ \exp\left\{\frac{xt}{1+t}\right\} f_n^{\beta+n}\left(\frac{x}{1+t}\right)=\sum_{p=0}^{\infty}\frac{(n+1)_p}{p!}\ f_{n+p}^{\beta+n-p}(x)\ t^p, \tag{2.6}$$

Which is note worthy?

Now we shall prove the Theorem -1, by using the result (2.6)

3. PROOF OF THE THEOREM 1

Now the right side of (1.3)

$$= \sum_{n=0}^{\infty} t^n\ \sigma_n(x,u,z)$$

$$= \sum_{n=0}^{\infty} t^n \sum_{p=0}^{n} a_p \binom{n}{p} f_n^{\beta-n+2p}(x)\ g_p(u)\ z^{n-p}$$

[using (1.4)]

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{\infty} t^{n+p}\ a_p \binom{n+p}{p} f_n^{\beta-n+p}(x)\ g_p(u)\ z^n$$

$$= \sum_{p=0}^{\infty} t^p\ a_p\ g_p(u)\left(\sum_{n=0}^{\infty}\binom{n+p}{p} f_{n+p}^{\beta-n+p}(x)\ (zt)^n\right)$$

$$= \sum_{p=0}^{\infty} t^p a_p\ (1+tz)^{p+\beta-1}\ \exp\left\{\frac{txz}{1+tz}\right\} f_p^{\beta+p}\left(\frac{x}{1+tz}\right) g_p(u)$$

[using (2.6)]

$$= (1+tz)^{\beta-1}\ \exp\left\{\frac{txz}{1+tz}\right\} \sum_{p=0}^{\infty} a_p\ f_p^{\beta+p}\left(\frac{x}{1+tz}\right) g_p(u)\ \{(1+tz)t\}^p$$

$$= (1+tz)^{\beta-1}\ \exp\left\{\frac{tzx}{1+tz}\right\} G\left(\frac{x}{1+tz}, u, (1+tz)t\right).$$

[using (1.2)]

This completes the proof of the theorem.

Now, we would like to point it out that the Theorem-1 can be proved by the direct application of the operator R by using the method as discussed in [7].

Now consider the following generating relation

$$G(x,u,w)=\sum_{n=0}^{\infty} a_n\ f_n^{\beta+n}(x)\ g_n(u)\ w^n. \tag{3.1}$$

Replacing w by wy and multiplying both sides of (3.1) by z^β, we get

$$z^\beta\ G(x,u,wy)=\sum_{n=0}^{\infty} a_n\left(f_n^{\beta+n}(x) y^n\ z^\beta\right) g_n(u)\ w^n. \tag{3.2}$$

Now operating ($\exp(wR)$) on both sides of (3.2), we get

$$\left(\exp(wR)\right)\left(z^\beta\ G(x,u,wy)\right)=\left(\exp(wR)\right)\ \left(\sum_{n=0}^{\infty} a_n\left(f_n^{\beta+n}(x) y^n\ z^\beta\right) g_n(u)\ w^n\right). \tag{3.3}$$

The left sides of (3.3), with the help of (2.6), becomes

$$=\left(1-\frac{wy}{z^2}\right)^{\beta-1}\exp\left\{\frac{-(wxy)/z^2}{1-(wy)/z^2}\right\}z^{\beta}G\left(\frac{x}{\left(1-(wy)/z^2\right)},u\ ,\left(1-(wy)/z^2\right)wy\right). \qquad (3.4)$$

The right side of (3.3), with the help of (2.4), becomes

$$=\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}a_n\frac{w^{n+p}}{p!}(-1)^p\ (n+1)_p\ f_{n+p}^{\beta+n-p}(x)\ g_n(u)\ y^{n+p}\ z^{\beta-2p} \qquad (3.5)$$

$$=z^{\beta}\sum_{n=0}^{\infty}(wy)^n\sum_{p=0}^{n}a_{n-p}\frac{(-1)^p\ (n-p+1)_p}{p!}f_n^{\beta+n-2p}(x)\ g_{n-p}(u)\ \ z^{-2p}\ .$$

Equating (3.4) and (3.5) and then putting $wy = t$ and $-z^{-2} = z$, we get

$$(1+tz)^{\beta-1}\ \exp\left\{\frac{tzx}{1+tz}\right\}G\left(\frac{x}{t+tz},u,(1+tz)t\right)=\sum_{n=0}^{\infty}t^n\sigma_n(x,u,z)$$

where

$$\sigma_n(x,u,z)=\sum_{p=0}^{n}a_p\binom{n}{p}f_n^{\beta-n+2p}(x)\ g_p(u)\ z^{n-p},$$

which is the Theorem 1 .

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