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ORIGINAL PAPER

CONDITIONAL CAUCHY EQUATIONS OF $T_{1,2}$-TYPE ON n-GROUPS

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Abstract. *J. Dhombres [2] made a classification of conditional Cauchy equations on groups. In [9] we extended the results obtained by J. Dhombres and R. Ger [3], [4] on conditional Cauchy equations of $T_{1,1}$-type (right cilinder type), to the similar equations on n-groups. In this paper we extend the results for conditional Cauchy equations of $T_{1,2}$-type.*

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1. INTRODUCTION

One of the conditional Cauchy equations on groups is the functional equation solved by J. Dhombres:

$$\begin{cases} f: G \to G \\ f(x * f(y)) = f(x) * f(f(y)), \ x, y \in G \end{cases} \tag{1}$$

where $(G,*)$ is a group.

The equation (1) is a conditional Cauchy equation with the conditioner $C(f) = G \times \operatorname{Im} f$.

The subject of this paper is the extention of the results obtained by J. Dhombres to the similar equation of n-groups:

$$\begin{cases} f: G \to G \\ f(\varphi(x, f(y_1), \dots, f(y_n))) = \varphi(f(x), f(f(y_1)), \dots, f(f(y_n))) \end{cases} \tag{2}$$

$x, y_1, \dots, y_n \in G$, where (G,φ) is an $(n+1)$-group with the $(n+1)$-ary operation $\varphi: G^{n+1} \to G$.

2. PRELIMINARY RESULTS

- If (G,φ) is an $(n+1)$-group and $Z \subset G^{n+1}$ is a fixed set, then the functional equation:

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$$\begin{cases} f: G \to G \\ f(\varphi(z_1, z_2, ..., z_{n+1})) = \varphi(f(z_1), f(z_2), ..., f(z_{n+1})), \end{cases} \quad (3)$$

$(z_1, z_2, ..., z_{n+1}) \in Z$, is called Z-functional Cauchy equation [6].

- A function $c: G^G \to P(G^{n+1})$ is called conditioner on G and if we denote $c(f) = Z_f \subset G^{n+1}$, $f \in G^G$, the functional equation

$$\begin{cases} f: G \to G \\ f(\varphi(z_1, z_2, ..., z_{n+1})) = \varphi(f(z_1), f(z_2), ..., f(z_{n+1})) \end{cases} \quad (4)$$

$(z_1, z_2, ..., z_{n+1}) \in Z_f$, is called conditional Cauchy equation, conditioned by the conditioner c [7].

Remark 2.1. The functional equation (1) is a conditional Cauchy equation in which the conditioner is

$$c: G^G \to P(G^{n+1}), \quad c(f) = G \times (\mathrm{Im}\, f)^n, \quad f \in G^G.$$

We recall some results which we will use in the following.

Theorem 2.2. [7] For every conditioner $c: G^G \to P(G^{n+1})$ there exists a family $\{Z_i \subset G_{n+1} \mid i \in I\}$ such that the set solution of functional equation (4) is the union of the set solutions of Z_i-functional Cauchy equations of form (3).

Theorem 2.3. [1], [9] A function $f: G \to G$ is a solution of the functional equation

$$\begin{cases} f: G \to G \\ f(\varphi(x, y_1, ..., y_n)) = \varphi(f(x), f(y_1), ..., f(y_n)) \end{cases} \quad (5)$$

$x \in G$, $y_1, ..., y_n \in Y \subset G$ (right cylinder type or $T_{1,1}$-type) iff f is a solution of the equation

$$\begin{cases} f: G \to G \\ f(x \circ y) = f(x) * f(y), \ x \in G, \ y \in G_0 \end{cases} \quad (6)$$

where $(G, \circ)$ and $(G, *)$ are the reduced Hosszu group

$$(G, \circ) = \mathrm{Red}_u(G, \varphi), \quad (G, *) = \mathrm{Red}_{f(u)}(G, \varphi),$$

with $u \in Y$, G_0 is the sub-$(n+1)$-group generated by Y in (G, φ) and the restriction $f: G_0 \to G$ is a $(n+1)$-group morphism.

Remark 2.4. [8] • The group operations "$\circ$" and "$*$" are defined by:

$$x \circ y = \varphi(x, \underset{n-2}{u}, \bar{u}, y), \quad x, y \in G$$

$$x * y = \varphi(x, \underset{n-2}{f(u)}, \overline{f(u)}, y), \quad x, y \in G$$

where $\bar{x}$ is the skew element of x in (G, φ).

• $(G_0, \circ)$ is a subgroup in $(G, \circ)$.

Theorem 2.5. [9] The general solution of equation (5) is:

$$f(x) = g(x \circ (s(p(x)))^{-1}) * h(p(x)),$$

where:

- G_0 is an arbitrary subgroup in G
- $p: G \to G/\rho$ is the canonical projection on the quotient set with respect to the equivalence relation $x\rho y \Leftrightarrow x \circ y^{-1} \in G_0$
- u is arbitrary in G_0
- $h: G/\rho \to G$ is an arbitrary function such that $h(p(u)) = g(u)$
- $s: G/\rho \to G$ is a lifting relative to p $(p \circ s \circ p = p)$
- $g: G_0 \to G$ is a morphism such that $g(s(p(1))) = 1$.

Theorem 2.6. [10] The Z-conditional Cauchy equation with the conditioner $Z = G \times G \times Y_3 \times ... \times Y_{n+1}$ is redundant for every sets $Y_3, ..., Y_n, Y_{n+1}$ with the property $Y_3 \cap ... \cap Y_n \cap Y_{n+1} \neq \varnothing$.

3. MAIN RESULT

Theorem 3.1. The function $f: G \to G$ is a solution of the conditional Cauchy equation (2) iff f has the form:

$$f(x) = g(x \circ (s(p(x)))^{-1}) \circ (g(u))^{-1} \circ h(p(x)), \quad x \in G$$

where:

- (G_0, φ) is an arbitrary sub-$(n+1)$-group in (G, φ)
- $(G, \circ)$ is the reduced Hosszu group $(G, \circ) = \mathrm{Red}_u(G, \varphi)$, with $u \in G_0$
- $g: G_0 \to G_0$ is an arbitrary $(n+1)$-group morphism
- $p: G \to G/\rho$ is the canonical projection with respect to the equivalence relation $x\rho y \Leftrightarrow x \circ y^{-1} \in G_0$
- $s: G \to G/\rho$ is an arbitrary lifting relative to p
- $h: G/\rho \to G_0$ is an arbitrary function with $h(p(u)) = g(u)$.

Proof: From Remark 2.1, the equation (2) is a conditional equation conditioned by the conditioner $c(f) = C \times Y_f^n$, where $Y_f = \mathrm{Im}\, f$. From Theorem 2.3 [9] for a fixed $Y = Y_f$, the equation (5) is equivalent with the Z-equation with $Z = G \times G_0^n$ and its solution is given by Theorem 2.5. Since the set $Y_f = \mathrm{Im}\, f$ is an arbitrary set, then G_0 is an arbitrary sub-$(n+1)$-group. Restriction of f to G_0 is g. Using the relations of the Hosszu reduced group by u and $f(u)$ [8] we have: $x * y = x \circ (g(u))^{-1} \circ y$ and the expression of f from Theorem 2.5 becomes the expression of Theorem 3.1. The other elements that appear have been defined in Theorem 2.5.

Theorem 3.1 can be rewritten without using the reduction to bigroups [5]. It is sufficient to take into account the relation between reduces and extendings.

Theorem 3.2. The function $f:G\to G$ is a solution of equation (2) iff there exists a sub-$(n+1)$-group (G_0,φ) in (G,φ), an element $u\in G_0$, a $(n+1)$ group morphism $g:G_0\to G_0$ such that:

$$f(x)=\varphi(g(\varphi(x,s(\underset{n-2}{p}(x)),\overline{s(p(x))},u),\underset{n-2}{g(u)},\overline{g(u)},h(p(x)))),\quad x\in G,$$

where p,s,h are those from Theorem 3.1.

If we consider the conditioner $c(f)=G\times G\times(\operatorname{Im} f)^{n-1}$ we obtain the functional equation

$$f(\varphi(x_1,x_2,f(x_3),...,f(x_{n+1})))=\varphi(f(x_1),f(x_2),f^2(x_3),...,f^2(x_{n+1})),\quad x_1,x_2,...,x_{n+1}\in G. \quad (7)$$

Using Theorem 2.6 we obtain:

Theorem 3.3. The function $f:G\to G$ is a solution of equation (7) if and only if it is a morphism.

Remark 3.4. Theorem 3.3 can be reformulated as: the equation (7) is redundant (only solutions are morphisms).

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