ORIGINAL PAPER

PARALLEL SURFACES TO S-TANGENT SURFACES OF BIHARMONIC S-CURVES ACCORDING TO SABBAN FRAME IN HEISENBERG GROUP HEIS^3

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Abstract. *In this paper, we study parallel surfaces to* $\mathsf{S}-$*tangent surfaces according to Sabban frame in the Heisenberg group Heis*^{3}. *We characterize parallel surfaces to* $\mathsf{S}-$*tangent surfaces of the biharmonic curves in terms of their geodesic curvature in the Heisenberg group Heis*^{3}. *Finally, we find explicit parametric equations of parallel surfaces to* $\mathsf{S}-$*tangent surfaces according to Sabban Frame.*

Mathematics Subject Classifications: *53C41, 53A10.*

Keywords: *Biharmonic curve, Heisenberg group, Tangent surface.*

1. INTRODUCTION

A parallel surface or an offset surface if you prefer is a surface for which the points on one surface are equidistant to a corresponding point in another one and the derivative at that point is the same on both surfaces.

In this paper, we study parallel surfaces to $\mathsf{S}-$tangent surfaces according to Sabban frame in the Heisenberg group Heis^3. We characterize parallel surfaces to $\mathsf{S}-$tangent surfaces of the biharmonic curves in terms of their geodesic curvature in the Heisenberg group Heis^3. Finally, we find explicit parametric equations of parallel surfaces to $\mathsf{S}-$tangent surfaces according to Sabban Frame.

2. PRELIMINARY RESULTS

Heisenberg group Heis^3 can be seen as the space R^3 endowed with the following multipilcation:

$$(\overline{x},\overline{y},\overline{z})(x,y,z)=(\overline{x}+x,\overline{y}+y,\overline{z}+z-\frac{1}{2}\overline{x}y+\frac{1}{2}x\overline{y}) \tag{2.1}$$

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Heis ^{3} is a three-dimensional, connected, simply connected and 2-step nilpotent Lie group.

The Riemannian metric g is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2.$$

The Lie algebra of Heis ^{3} has an orthonormal basis

$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0$$

with

$$g(e_1, e_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1.$$

we obtain

$$\begin{aligned} \nabla_{\mathbf{e}_1}\mathbf{e}_1 &= \nabla_{\mathbf{e}_2} e_2 = \nabla_{\mathbf{e}_3}\mathbf{e}_3 = 0, \\ \nabla_{\mathbf{e}_1}\mathbf{e}_2 &= -\nabla_{\mathbf{e}_2}\mathbf{e}_1 = \frac{1}{2}\mathbf{e}_3, \\ \nabla_{\mathbf{e}_1}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_1 = -\frac{1}{2}\mathbf{e}_2, \\ \nabla_{\mathbf{e}_2}\mathbf{e}_3 &= \nabla_{\mathbf{e}_3}\mathbf{e}_2 = \frac{1}{2}\mathbf{e}_1. \end{aligned} \tag{2.3}$$

The components $\{R_{ijkl}\}$ of R relative to $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ are defined by

$$R_{ijk} = R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_k, \ R_{ijkl} = R(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k, \mathbf{e}_l) = g\big(R(\mathbf{e}_i, \mathbf{e}_j)\mathbf{e}_l, \mathbf{e}_k\big).$$

The non vanishing components of the above tensor fields are

$$R_{121} = \frac{3}{4}\mathbf{e}_2, \quad R_{131} = -\frac{1}{4}\mathbf{e}_3, \quad R_{122} = -\frac{3}{4}\mathbf{e}_1,$$

$$R_{232} = -\frac{1}{4}\mathbf{e}_3, \quad R_{133} = \frac{1}{4}\mathbf{e}_1, \quad R_{233} = \frac{1}{4}\mathbf{e}_2,$$

and

$$R_{1212} = -\frac{3}{4}, \quad R_{1313} = R_{2323} = \frac{1}{4}.$$

3. BİHARMONİC S-CURVES ACCORDİNG TO SABBAN FRAME IN THE HEİSENBERG GROUP HEİS ^{3}

Let $\gamma : I \to Heis^3$ be a non geodesic curve on the Heisenberg group Heis ^{3} parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis ^{3} along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N},$$
$$\nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad (3.1)$$
$$\nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N},$$

where κ is the curvature of γ and τ is its torsion,

$$g(\mathbf{T},\mathbf{T}) = 1, g(\mathbf{N},\mathbf{N}) = 1, g(\mathbf{B},\mathbf{B}) = 1,$$
$$g(\mathbf{T},\mathbf{N}) = g(\mathbf{T},\mathbf{B}) = g(\mathbf{N},\mathbf{B}) = 0.$$

Now we give a new frame different from Frenet frame. Let $\alpha : I \to \mathbb{S}^2_{Heis^3}$ be unit speed spherical curve. We denote σ as the arc-length parameter of α . Let us denote $\mathbf{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of α. We now set a vector $\mathbf{s}(\sigma) = \alpha(\sigma) \times \mathbf{t}(\sigma)$ along α. This frame is called the Sabban frame of α on the Heisenberg group $Heis^3$. Then we have the following spherical Frenet-Serret formulae of α :

$$\nabla_{\mathbf{t}}\alpha = \mathbf{t},$$
$$\nabla_{\mathbf{t}}\mathbf{t} = -\alpha + \kappa_g \mathbf{s}, \qquad (3.2)$$
$$\nabla_{\mathbf{t}}\mathbf{s} = -\kappa_g \mathbf{t},$$

where κ_g is the geodesic curvature of the curve α on the $\mathbb{S}^2_{Heis^3}$ and

$$g(\mathbf{t},\mathbf{t}) = 1, g(\alpha,\alpha) = 1, g(\mathbf{s},\mathbf{s}) = 1,$$
$$g(\mathbf{t},\alpha) = g(\mathbf{t},\mathbf{s}) = g(\alpha,\mathbf{s}) = 0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3,$$
$$\mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3, \qquad (3.3)$$
$$\mathbf{s} = s_1\mathbf{e}_1 + s_2\mathbf{e}_2 + s_3\mathbf{e}_3.$$

To separate a biharmonic curve according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $\mathbb{S}$-curve.

Theorem 3.1. ([9]) *Let* $\alpha : I \to \mathbb{S}^2_{Heis^3}$ *be a unit speed non-geodesic biharmonic* $\mathbb{S}$-*curve. Then, the parametric equations of* α *are*

$$x(\sigma) = -\frac{\sin^2 \mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2,$$
$$y(\sigma) = \frac{\sin^2 \mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_3, \qquad (3.4)$$
$$z(\sigma) = \cos \mathsf{E}\,\sigma - \frac{\mathsf{V}\sigma + \mathsf{M}_1}{2\mathsf{V}^2}\sin^4 \mathsf{E} - \frac{\sin 2[\mathsf{M}\sigma + \mathsf{M}_1]}{4\mathsf{V}^2}\sin^4 \mathsf{E}$$
$$+ \frac{\mathsf{M}_2}{\mathsf{V}}\sin^3 \mathsf{E}\sin[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_4,$$

where $\mathsf{M}_1, \mathsf{M}_2, \mathsf{M}_3, \mathsf{M}_4$ *are constants of integration and*

$$\mathsf{M} = \left(\frac{\sqrt{1+\kappa_g^2}}{\sin \mathsf{E}} - \cos \mathsf{E}\right) \text{ and } \mathsf{V} = \sqrt{1+\kappa_g^2} - \frac{1}{2}\sin 2\mathsf{E}.$$

We can use Mathematica in above theorem, yields

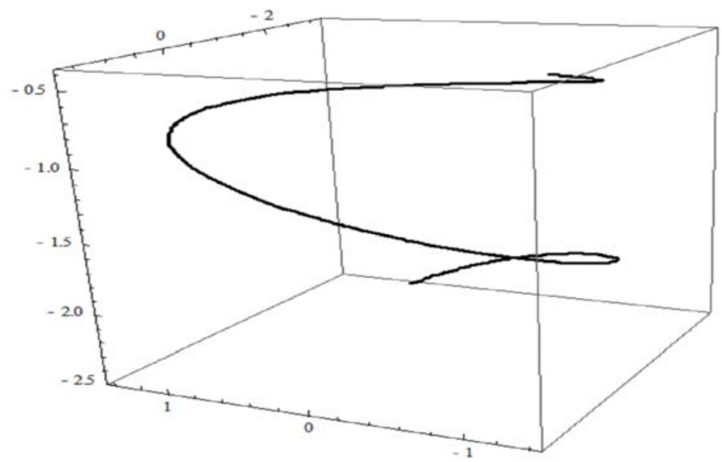


Fig. 1.

4. PARALLEL SURFACES TO $\mathsf{S}-$TANGENT SURFACES OF BIHARMONIC S-CURVES ACCORDING TO SABBAN FRAME IN THE HEISENBERG GROUP HEIS^3

To separate a tangent surface according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for this surface as $\mathsf{S}-$tangent surface.

The purpose of this section is to study parallel surfaces to $\mathsf{S}-$tangent surfaces of biharmonic S-curve in the Heisenberg group Heis^3.
The $\mathsf{S}-$tangent surface of γ is a ruled surface

$$\mathsf{R}^{\mathsf{S}}(\sigma,u)=\alpha(\sigma)+u\alpha'(\sigma). \tag{4.1}$$

Theorem 4.1. *Let* R^{S} *be a* $\mathsf{S}-$*tangent surface of a unit speed non-geodesic biharmonic* $\mathsf{S}-$*curve in the Heisenberg group Heis*^{3}. *Then, the parametric equations of* $\mathsf{S}-$*tangent surface of* α *are*

$$\begin{aligned}
\mathbf{x}_{\mathsf{R}^{\mathsf{S}}}(\sigma,u)&=-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+u\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2,\\
\mathbf{y}_{\mathsf{R}^{\mathsf{S}}}(\sigma,u)&=\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+u\sin\mathsf{E}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3,\\
\mathbf{z}_{\mathsf{R}^{\mathsf{S}}}(\sigma,u)&=\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}\\
&+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+u[\cos\mathsf{E}+\sin\mathsf{E}(-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]\\
&+\mathsf{M}_2)\cos[\mathsf{M}\sigma+\mathsf{M}_1]]+\mathsf{M}_4,
\end{aligned} \tag{4.2}$$

where $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3,\mathsf{M}_4$ *are constants of integration and*

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E})\text{and}\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E}.$$

A parallel surface to $S-$tangent surface of γ is a parametrized surface

$$A(s,u)=R^{S}(s,u)+\hat{o}\mathbf{N}_{R^{S}}(s,u), \tag{4.3}$$

where $\hat{o}$ is a constant.

Firstly, we need following lemma.

Lemma 4.2. *Let* R^{S} *be a* $S-$*tangent surface of a unit speed non-geodesic biharmonic* $S-$*curve in the Heisenberg group Heis*^{3}. *Then, the normal vector of* $S-$*tangent surface of* α *is*

$$\begin{aligned}
\mathbf{N}_{R^{S}} &= [-u\kappa_g[-\frac{\sin^2 E}{V}\cos[M\sigma+M_1]+M_2]-\frac{u}{\kappa_g}[\sin E\cos[M\sigma+M_1](M+\cos E)\\
&-\frac{\sin^2 E}{V}\cos[M\sigma+M_1]+M_2]]\mathbf{e}_1+[-u\kappa_g[\frac{\sin^2 E}{V}\sin[M\sigma+M_1]+M_3]\\
&-\frac{u}{\kappa_g}[-\sin E\sin[M\sigma+M_1](M+\cos E)+\frac{\sin^2 E}{V}\sin[M\sigma+M_1]+M_3]]\mathbf{e}_2\\
&+[-u\kappa_g[\cos E\sigma-\frac{V\sigma+M_1}{2V^2}\sin^4 E-\frac{\sin 2[M\sigma+M_1]}{4V^2}\sin^4 E\\
&-[\frac{\sin^2 E}{V}\sin[M\sigma+M_1]+M_3][-\frac{\sin^2 E}{V}\cos[M\sigma+M_1]+M_2]\\
&+\frac{M_2}{V}\sin^3 E\sin[M\sigma+M_1]+M_4]-\frac{u}{\kappa_g}[\cos E\sigma-\frac{V\sigma+M_1}{2V^2}\sin^4 E\\
&-\frac{\sin 2[M\sigma+M_1]}{4V^2}\sin^4 E-[\frac{\sin^2 E}{V}\sin[M\sigma+M_1]+M_3][-\frac{\sin^2 E}{V}\cos[M\sigma+M_1]\\
&+M_2]+\frac{M_2}{V}\sin^3 E\sin[M\sigma+M_1]+M_4]]\mathbf{e}_3,
\end{aligned} \tag{4.4}$$

where

$$M=(\frac{\sqrt{1+\kappa_g^2}}{\sin E}-\cos E)\text{and}V=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2E.$$

Proof: We can easily verify that

$$\nabla_{\mathbf{t}}\mathbf{t}=(t_1^{'}+t_2t_3)\mathbf{e}_1+(t_2^{'}-t_1t_3)\mathbf{e}_2+t_3^{'}\mathbf{e}_3. \tag{4.5}$$

Since, we immediately arrive at

$$\nabla_{\mathbf{t}}\mathbf{t}=\sin E\cos[M\sigma+M_1](M+\cos E)\mathbf{e}_1-\sin E\sin[M\sigma+M_1](M+\cos E)\mathbf{e}_2.$$

Obviously, we also obtain

$$\begin{aligned}
s(\sigma) &= \frac{1}{\kappa_g}[\sin E\cos[M\sigma+M_1](M+\cos E)-\frac{\sin^2 E}{V}\cos[M\sigma+M_1]+M_2]\mathbf{e}_1\\
&+\frac{1}{\kappa_g}[-\sin E\sin[M\sigma+M_1](M+\cos E)+\frac{\sin^2 E}{V}\sin[M\sigma+M_1]+M_3]\mathbf{e}_2\\
&+\frac{1}{\kappa_g}[\cos E\sigma-\frac{V\sigma+M_1}{2V^2}\sin^4 E-\frac{\sin 2[M\sigma+M_1]}{4V^2}\sin^4 E\\
&-[\frac{\sin^2 E}{V}\sin[M\sigma+M_1]+M_3][-\frac{\sin^2 E}{V}\cos[M\sigma+M_1]+M_2]
\end{aligned} \tag{4.6}$$

$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4]\mathbf{e}_3,$$

where

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E})\text{and}\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E}.$$

Now, we can prove the following interesting main result.

Theorem 4.3. *Let* R^{S} *be a* $\mathsf{S}-$*tangent surface of a unit speed non-geodesic biharmonic* $\mathsf{S}-$*curve in the Heisenberg group Heis*^{3}. *Then, equation of parallel surface to* $\mathsf{S}-$*tangent surface of* α *is*

$$\mathbf{A}_{\mathsf{R}^{\mathsf{S}}}=[[[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+u\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$

$$-\hat{o}u\kappa_g[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]-\frac{\hat{o}u}{\kappa_g}[\sin\mathsf{E}\cos[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})$$

$$-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]]]\mathbf{e}_1+[[[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]$$

$$+u\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]-\hat{o}u\kappa_g[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]$$

$$-\frac{\hat{o}u}{\kappa_g}[-\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})+\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]]]\mathbf{e}_2$$

$$+[[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]$$

$$[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]+[-\hat{o}u\kappa_g[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}$$

$$-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$

$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+u\cos\mathsf{E}+\mathsf{M}_4]+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4]$$

$$-\frac{\hat{o}u}{\kappa_g}[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]$$

$$-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4]]]\mathbf{e}_3,$$

where $\mathsf{S}_1,\mathsf{S}_2,\mathsf{S}_3,\mathsf{S}_4,\mathsf{S}_5$ *are constants of integration·*

Proof: By using (4.2) and (4.3) we obtain (4.4). Hence the proof is completed.

Theorem 4.4. *Let* R^{S} *be a* $\mathsf{S}-$*tangent surface of a unit speed non-geodesic biharmonic* $\mathsf{S}-$*curve in the Heisenberg group Heis*^{3}. *Then, parametric equations of parallel surface to* $\mathsf{S}-$*tangent surface of* α *are*

$$\mathbf{x}=[[[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+u\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]-\hat{o}u\kappa_g[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$

$$-\frac{\hat{o}u}{\kappa_g}[\sin\mathrm{E}\cos[\mathrm{M}\sigma+\mathrm{M}_1](\mathrm{M}+\cos\mathrm{E})-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]],$$

$$\mathbf{y}=[[[-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+u\sin\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]-\hat{o}u\kappa_g[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]$$

$$-\frac{\hat{o}u}{\kappa_g}[-\sin\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1](\mathrm{M}+\cos\mathrm{E})+\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]]],$$

$$\mathbf{z}=[[[-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+u\sin\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]-\hat{o}u\kappa_g[-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]$$

$$-\frac{\hat{o}u}{\kappa_g}[\sin\mathrm{E}\cos[\mathrm{M}\sigma+\mathrm{M}_1](\mathrm{M}+\cos\mathrm{E})-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]]]$$

$$[[[-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+u\sin\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]-\hat{o}u\kappa_g[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]$$

$$-\frac{\hat{o}u}{\kappa_g}[-\sin\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1](\mathrm{M}+\cos\mathrm{E})+\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]]]$$

$$+[[\cos\mathrm{E}\sigma-\frac{\mathrm{V}\sigma+\mathrm{M}_1}{2\mathrm{V}^2}\sin^4\mathrm{E}-\frac{\sin 2[\mathrm{M}\sigma+\mathrm{M}_1]}{4\mathrm{V}^2}\sin^4\mathrm{E}-[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]$$

$$[-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]+[-\hat{o}u\kappa_g[\cos\mathrm{E}\sigma-\frac{\mathrm{V}\sigma+\mathrm{M}_1}{2\mathrm{V}^2}\sin^4\mathrm{E}-\frac{\sin 2[\mathrm{M}\sigma+\mathrm{M}_1]}{4\mathrm{V}^2}\sin^4\mathrm{E}$$

$$-[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3][-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]+\frac{\mathrm{M}_2}{\mathrm{V}}\sin^3\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]$$

$$+u\cos\mathrm{E}+\mathrm{M}_4]+\frac{\mathrm{M}_2}{\mathrm{V}}\sin^3\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_4]-\frac{\hat{o}u}{\kappa_g}[\cos\mathrm{E}\sigma-\frac{\mathrm{V}\sigma+\mathrm{M}_1}{2\mathrm{V}^2}\sin^4\mathrm{E}$$

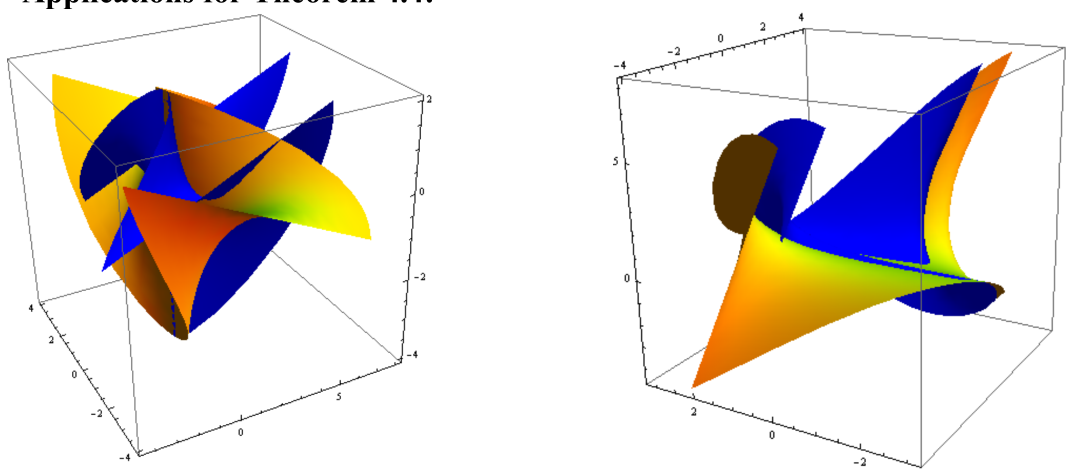
$$-\frac{\sin 2[\mathrm{M}\sigma+\mathrm{M}_1]}{4\mathrm{V}^2}\sin^4\mathrm{E}-[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3][-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]$$

$$+\frac{\mathrm{M}_2}{\mathrm{V}}\sin^3\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_4]]],$$

where $\mathrm{S}_1,\mathrm{S}_2,\mathrm{S}_3,\mathrm{S}_4,\mathrm{S}_5$ *are constants of integration·*

Proof: Omitted.

Applications for Theorem 4.4:



Figs. 2&3. The equations of tangent developable and its parallel surface are illustrated colour Blue, Yellow' respectively.

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