ORIGINAL PAPER

SOME CONSEQUENCES OF THE BROWDER'S THEOREM IN HILBERT SPACES

CORNELIU UDREA[1]

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Abstract. *This work deals with the Browder's theorem in Hilbert spaces. One can remark that the theorems of Riesz, Stampacchia, and Lax-Milgram are consequences of the Browder's theorem. Moreover in a vector space with a scalar product the Browder's property is equivalent to the completness of the space. After that in the non-linear case a Stampacchia and Lax-Milgram type theorems are stated and proved.*

Keywords: *Browder's theorem, coercive map, monotone map*

1. INTRODUCTION

In this section we present the general setting of this work [1-3].

1.1 Remark. (i). We denote by $\langle X,Y\rangle$ a real dual system, and by $\langle\, ,\rangle_X : X\times Y\to\mathbb{R}$ the corresponding pairing. Moreover $\sigma(X,Y)$ (respectively $\sigma(Y,X)$) denotes the weak topology on *X* (respectively on *Y*) defined by *Y* (respectively *X*) [6].

(ii). We assume that *C* is a non empty convex, and weakly closed subset of *X*, and that *T* is a (nonlinear) function from *C* into *Y*.

1.2 Definition. (i). We shall say that *T* is *monotone* (respectively strictly monotone) on *C* if $\langle x-y,Tx-Ty\rangle_X\ge 0$ (respectively $\langle x-y,Tx-Ty\rangle_X>0$) for all $x,y\in C,\ x\ne y$.

(ii). The function *T* is called *weakly continuous on the line segments* of *C* if the mapping $(c\mapsto\langle x,Tc\rangle):[a,b]\to\mathbb{R}$ is continuous for all $x\in X$, and $a,b\in C$.

1.3. Remark. (i). Let $a\in C$, and $C-a:=\{c-a:c\in C\}$. Obviously *C–a* is a convex and weakly closed subset of *X* which contains 0_E. If we define

$$T_a:C-a\to Y,\ T_a(c-a):=Tc,\ \forall c\in C,$$

and if we assume that *T* is monotone on *C*, and weakly continuous on the line segments of *C*, then the function T_a has the same properties on C_a.

(ii). We consider *E* a real normed space, and *E'* its topological dual endowed with the canonical norm. We shall distinguish the following dual systems. (a). $\langle E,E'\rangle$ where $\langle x,x'\rangle_E:=x'(x)$; (b). $\langle E',E\rangle$ where $\langle x',x\rangle_{E'}:=x'(x)$ for all $(x,x')\in E\times E'$.

[1] University of Pitești, Faculty of Mathematics and Computer Science, 110040 Pitesti, Romania.
E-mail: corneliu_udrea@yahoo.com; corneliu.udrea@upit.ro.

1.4 Definition. If X is a real normed space, and $b \in X$ then T is called *coercive on C with respect to b* if

$$\liminf_{c\in C,\, \|c\|\to\infty} \frac{\langle c-b, Tc\rangle}{\|c-b\|} = \infty.$$

When $b = 0_E$ we shall say that *T is coercive on C.*

1.5 Remark. In the case of Remark 1.3 (i) we have the following assertions.

(i). If T is coercive on C, then T_a is coercive on C_a with respect to $-a$.

(ii). If T is coercive on C with respect to b, the T_a is coercive on C_a with respect to $b-a$.

Therefore, from now on, we shall say that T is coercive on C (for example) instead of T is coercive on C with respect to a certain point.

1.6 Definition. Let X be a real normed space.

(i). A function T from C into Y which is monotone on C, coercive on C and weakly continuous on the line segments of C is called *a Browder-Minty operator on C*.

(ii). We shall say that a Browder-Minty operator on *C has the Browder's property* if it satisfies the following condition,

(B). $\forall y \in Y,\ \exists c_y \in C$ such that $\langle c_y - c, Tc_y - y\rangle_X \le 0,\ \forall c \in C.$

1.7 Theorem. (i). *If E is a reflexive Banach space, and C is a subset of E as in Remark 1.1 (ii), then all Browder-Minty operator from C into E' has the Browder's property (Browder's theorem,* [1, 3].

(ii). *For all Banach space E, and C a subset of E' as in Remark 1.1 (ii), all Browder-Minty operator from C into E has the Browder's property (a Browder type theorem,* [7].

1.8 Remark. (i). When the Browder-Minty operator T is strictly monotone on C, the element c_y (where $y \in Y$) is the unique element of C having the (B) - property.

(ii). Since every Hilbert space is a reflexive one we can apply in such a space the Browder's theorem.

2. LINEAR CONSEQUENCES

From now on $\mathbb{H}$ is a real vector space, and $\langle\ ,\ \rangle$ is a scalar product on $\mathbb{H}$. As usual $(\mathbb{H}', \|\cdot\|)$ denotes its topological dual which is a Banach space with respect to the canonical norm (denoted $\|\cdot\|$), and $\langle\ ,\ \rangle_{\mathbb{H}}$ is the pairing on $\mathbb{H}\times\mathbb{H}'$

2.1 Definition. For all h from $\mathbb{H}$ the mapping $(x \mapsto \langle x, h\rangle) : \mathbb{H} \to \mathbb{R}$ is a continuous linear functional on $\mathbb{H}$ with the norm equal to $\|h\|$. We denote by $\mathbb{I}$ the function

$$(h \mapsto \langle \cdot, h\rangle) : \mathbb{H} \to \mathbb{H}',$$

i.e. for all $h \in \mathbb{H}$, the element $\mathbb{I}h \in \mathbb{H}'$ is defined by

$$\langle x, \mathbb{I}h\rangle_{\mathbb{H}} = (\mathbb{I}h)(x) := \langle x, h\rangle.$$

2.2 Lema. *(i). The function* $\mathbb{I}:(\mathbb{H},\|\cdot\|)\to(\mathbb{H}',\|\cdot\|)$ *is an isometric linear operator.*

(ii). The operator $\mathbb{I}$ *is a Browder-Minty strictly monotone operator from* $\mathbb{H}$ *into* $\mathbb{H}$*'.*

Proof: (i). It is obvious.

(ii). Since $\mathbb{I}$ is a linear operator, and $\langle h,\mathbb{I}h\rangle_{\mathbb{H}}=\|h\|^2$ the assertion is obvious. □

2.3 Theorem. *The following assertions are equivalent.*

(i). The space $(\mathbb{H},\langle\ ,\ \rangle)$ *is a Hilbert space.*

(ii) All Browder-Minty operator from a non empty, convex and closed subset of $\mathbb{H}$ *into* $\mathbb{H}$*' has the Browder's property.*

(iii). The linear operator $\mathbb{I}$ *has the Browder's property.*

(iv). The operator $\mathbb{I}$ *is a bijective isometry.*

(v). The Riesz representation theorem is true for $\mathbb{H}$*' (i.e. for all* $h'\in\mathbb{H}'$ *there exists* $h\in\mathbb{H}$ *such that* $h'(x)=\langle x,h\rangle$ *for all* $x\in\mathbb{H}$*).*

Proof: (i)$\Rightarrow$(ii). It is the Remark 1.1.8. (ii).

(ii) $\Rightarrow$ (iii). In view of Lema 2.2.2 it is obvious.

(iii) $\Rightarrow$ (iv). By Remark 1.1.8. (i) and the hypothesis, for all $h'\in\mathbb{H}'$ there exists a unique element $h\in\mathbb{H}$ such that

$$\langle h-x,\mathbb{I}h-h'\rangle_{\mathbb{H}}\le 0,\ \forall x\in\mathbb{H}.$$

Since $\mathbb{H}$ is a vector space this means that

$$\langle x,\mathbb{I}h-h'\rangle_{\mathbb{H}}=0,\ \forall x\in\mathbb{H}\Leftrightarrow\langle\cdot,h\rangle=\mathbb{I}h=h'.$$

(iv) $\Rightarrow$ (v). It is obvious.

(v) $\Rightarrow$ (i). By Lemma 2.2.2, and the Riesz representation theorem for $\mathbb{H}'$ it results directly that $\mathbb{I}:(\mathbb{H},\|\cdot\|)\to(\mathbb{H}',\|\cdot\|)$ is a bijective, bounded, and linear operator, hence $\mathbb{I}$ is an isomorphism from $(\mathbb{H},\|\cdot\|)$ onto $(\mathbb{H}',\|\cdot\|)$. Since $(\mathbb{H}',\|\cdot\|)$ is a Banach space we have that $(\mathbb{H},\langle\ ,\ \rangle)$ is a Hilbert space. □

2.4 Remark. If $\mathbb{H}$ is a Hilbert space, and C is a nonempty convex, and closed subset of $\mathbb{H}$, then i_c denotes the canonical imbedding of C into $\mathbb{H}$. It is obvious that i_c is a Browder-Minty operator, hence it has the Browder's property i.e.

$$\forall h\in\mathbb{H},\ \exists c_h\in C,\ \langle c_h-c,c_h-h\rangle=\langle c_h-c,i_C c_h-h\rangle\le 0,\ \forall c\in C.$$

This means that c_h is the projection of h into C [4].

2.5 Definition. Let φ be a real bilinear form on $\mathbb{H}$, and A a non-empty subset of $\mathbb{H}$. We shall say that φ is *coercive on A* if there exists $\gamma_1\in(0,\infty)$ such that

$$\varphi(a,a)\ge\gamma_1\|a\|^2,\ \forall a\in A.$$

2.6 Theorem. *(Stampacchia's theorem, [1, 2]). We consider C a non-empty, convex, closed subset of a real Hilbert space* $\mathbb{H}$*, and* φ *a real bilinear form on the space* $(sp_{\mathbb{R}}C)\times\mathbb{H}$ *which is coercive on C, and separately continuous on* $C\times\mathbb{H}$*. Then for all* $h\in\mathbb{H}$ *there exists a unique element* c_h *from C such that*

$$\varphi(c_h, c_h - c) \le \langle h, c_h - c\rangle,\ \forall c\in C.$$

Proof: For all $c\in C$ the mapping

$$(h\mapsto\varphi(c,h)):\mathbb{H}\to\mathbb{R}$$

is a linear continuous functional, hence there exists a unique element from $\mathbb{H}$ (denoted by T_c) such that

$$\varphi(c,h)=\langle Tc,h\rangle,\ \forall h\in\mathbb{H}$$

Obviously the mapping T from C into $\mathbb{H}$ is weakly continuous (where C is endowed with the norm topology of $\mathbb{H}$). Moreover for all $c\in C$

$$\langle Tc,c\rangle=\varphi(c,c)\ge\gamma_1\|c\|^2,$$

hence T is coercive on C, and for all $c_1,c_2\in C$

$$\langle Tc_1-Tc_2,c_1-c_2\rangle=\varphi(c_1-c_2,c_1-c_2)\ge\gamma_1\|c_1-c_2\|^2$$

which means that T is strictly monotone.

In view of the Browder's property for all $h\in\mathbb{H}$ there exists a unique element $c_h\in C$ such that

$$\forall c\in C,\ \langle Tc_h-h,c_h-c\rangle\le 0\Leftrightarrow\forall c\in C,\ \varphi(c_h,c_h-c)=\langle Tc_h,c_h-c\rangle\le\langle h,c_h-c\rangle.\quad\square$$

2.7 Remark. We recall that a real bilinear form, φ, on $\mathbb{H}$ is continuous if there exists $\gamma_2\in(0,\infty)$ such that

$$\forall x,y\in\mathbb{H},\ |\varphi(x,y)|\le\gamma_2\|x\|\cdot\|y\|.$$

2.8 Theorem. (Lax-Milgram's theorem, [4]). *Let* $\mathbb{H}$ *be a real Hilbert space, and* φ *a real bilinear, continuous, and coercive form on* $\mathbb{H}$*. There exists a unique linear, bounded operator* T_φ *on* $\mathbb{H}$ *such that*

$$\varphi(x,y)=\langle T_\varphi x,y\rangle,\ \forall x,y\in\mathbb{H}.$$

Moreover T_φ *is an invertible operator.*

Proof: As in the proof of Theorem 2.2.6 we define T_φ a linear operator on $\mathbb{H}$ with the following property

$$\varphi(x,h)=\langle T_\varphi x,h\rangle,\ \forall x,h\in\mathbb{H}.$$

For all $h \in \mathbb{H}$ we have

$$\|T_\varphi x\| = \|\varphi(x,\cdot)\| = \sup_{h\in\mathbb{H},\|h\|\le 1} \varphi(x,h) \le \gamma_2 \|x\|,$$

hence T_φ is a bounded linear operator on $\mathbb{H}$.

Moreover since

$$\langle T_\varphi x - T_\varphi y, x-y\rangle = \varphi(x-y, x-y) \ge \gamma_1 \|x-y\|^2,$$

T_φ *i*s a coercive, and strictly monotone operator on $\mathbb{H}$. Hence, in view of the Browder's property, T_φ is a surjective map and by the strict monotony it is also injective.

The uniqueness of T_φ is obvious. □

3. NON-LINEAR APPLICATIONS

In this section $\mathbb{H}$ is a real Hilbert space, and C is a non-empty, convex and closed subset of $\mathbb{H}$.

3.1 Remark. As in [2] we shall consider from now on a function $\varphi : C \times \mathbb{H}$ into $\mathbb{R}$ which satisfies the following conditions.

(L1) (a). For all $c \in C$, $\varphi(c,\cdot)$ is a (weakly) continuous linear functional on $\mathbb{H}$.

(b). For all $h \in \mathbb{H}$, $\varphi(\cdot,h)$ is continuous on the line segments of C.

(L2) There exists $\delta \in (0,\infty)$, and $p \in (1,\infty)$ such that $\varphi(c,c) \ge \delta \|c\|^p$, $\forall c \in C$.

(L3) For all $c_1, c_2 \in C$, $c_1 \ne c_2$ we have that $\varphi(c_1, c_1 - c_2) - \varphi(c_2, c_1 - c_2) \ge 0$.

3.2 Theorem. (a Stampacchia type theorem). *For all $h \in \mathbb{H}$ there exists $c_h \in C$ such that $\varphi(c_h, c_h - c) \le \langle h, c_h - c\rangle$, $\forall c \in C$.*

Proof: Since for all $c \in C$, $\varphi(c,\cdot) \in \mathbb{H}'$, there exists a unique element from $\mathbb{H}$ (denoted by Tc) with the property $\forall h \in \mathbb{H}, \varphi(c,h) = \langle Tc, h\rangle$.

In view of the condition *(L3)* the function T from C into $\mathbb{H}$ has the following property: for all $c_1, c_2 \in C$, $\langle Tc_1 - Tc_2, c_1 - c_2\rangle = \varphi(c_1, c_1 - c_2) - \varphi(c_2, c_1 - c_2) \ge 0$.

i.e. T is monotone on C.

Moreover by *(L2)*, for all $c \in C$ $\langle Tc, c\rangle = \varphi(c,c) \ge \delta \|c\|^p$,

Hence T is coercive on C.

According to the condition *(L1) (b)*

$$(c \mapsto \langle Tc, h\rangle) = (c \mapsto \varphi(c,h)) : C \to \mathbb{R}$$

is continuous on the line segments of C, and this means that T is weakly continuous on the line segments of C.

Therefore, T is a Browder-Minty operator from C into $\mathbb{H}$, hence it has the Browder's property, i.e. $\forall h \in \mathbb{H}, \exists c_h \in C$ such that $\langle Tc_h - h, c_h - c\rangle \le 0$, $\forall c \in C$.

We have that

$$\varphi(c_h, c_h - c) = \langle Tc_h, c_h - c \rangle \leq \langle h, c_h - c \rangle, \ \forall c \in C. \ \square$$

3.3 Corollary. *If the inequality from the condition (L_3) is a strict one, then for all $h \in \mathbb{H}$ the element c_h from C which satisfies the relation of the previous theorem is unique.*

Proof: It is obvious because in this case *T* is strictly monotone. □

3.4 Theorem. (a Lax – Milgram type theorem). *Let C be equal to $\mathbb{H}$. Then there exists a unique (nonlinear) Browder-Minty operator from $\mathbb{H}$ into $\mathbb{H}$ such that*

$$\varphi(x, y) = \langle Tx, y \rangle, \ \forall x, y \in \mathbb{H}.$$

Proof: We apply the same steps as in the proof of the Theorem 3.3.2. □

3.5 Corollary. *Suppose that the inequality (L_3) is strict. Then the operator T of the previous theorem is a bijective function.*

Proof: By the Browder's property it is surjective, and in view of our hypothesis *T* is injective. □

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