ORIGINAL PAPER

SOLUTION OF TIME INDEPENDENT SCHRÖDINGER EQUATION FOR THE QUANTUM HARMONIC OSCILLATOR USING FRACTIONAL HARTLEY TRANSFORM

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Abstract. *In this paper we have proved some operation transform formulae for fractional Hartley transform in section 2. Solution of Time Independent Schrödinger Equation for the Quantum Harmonic Oscillator was found using the above operation transform formulae in section 3.*

Keywords: *Fractional Fourier transforms, Fractional Hartley transform.*

1. INTRODUCTION

The fractional integral transforms play an important role in signal processing. Fourier analysis is one of the most frequently used tools in signal processing and many other scientific disciplines.

Namias [2] introduced the concept of Fourier transform of fractional order, which depends on a continuous parameter α. The fractional Fourier transform with α = 1 corresponds to the classical Fourier transform and fractional Fourier transform with α = 0 corresponds to the identity operator. The fractional Fourier transforms and its properties were discussed in Ozaktas [3]. Bhosale and Chaudhary [1] had extended it to the distribution of compact support.

Using the eigenvalue function, as used in fractional Fourier transform, different integral transform in Fourier class that is cosine transform, sine transform and Hartley transform, are generalized to fractional transform by Pei [5]. For the generalization of fractional Hartley transform, he had shown that for all non negative integer m, $e^{\frac{-t^2}{2}} H_m(t)$ is the eigen function of the Hartley transform and had given the formula for fractional Hartley transform as,

$$H^{\alpha}\{f(t)\}(s) = \int_{-\infty}^{\infty} f(t) K_{\alpha}(t,s)dt,$$

where

$$K_{\alpha}(t,s) = \sqrt{\frac{1-i\cot\phi}{2\pi}}\, e^{i\frac{s^2}{2}\cot\phi} e^{i\frac{t^2}{2}\cot\phi} \frac{1}{2}\left[(1-ie^{i\phi})cas(\csc\phi.st) + (1+ie^{i\phi})cas(-\csc\phi.st)\right]$$

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In this paper first we have defined generalized fractional Hartley transform in section 2. We have proved some operation transform formulae for fractional Hartley transform in section 3. Solution of time independent Schrödinger equation for the quantum Harmonic oscillator is found using those operation transform formulae in section 4.

2 GENERALIZED FRACTIONAL HARTLEY TRANSFORM

2.1. THE TEST FUNCTION SPACE E(R^n)

An infinitely differentiable complex valued function ψ on R^n belongs to $E(R^n)$ if for each compact set $K \subset S_a$ where $S_a = \left\{ t \in R^n, \left| t \right| \leq a, a > 0 \right\}$,

$$\gamma_{E,k}(\psi) = \sup_{t \in K} \left| D_t^k \, \psi(t) \right| < \infty, \qquad k = 1,2,3.....$$

Note that the space E is complete and therefore a Frechet space.

2.2. THE FRACTIONAL HARTLEY TRANSFORM ON E'

It can be easily proved that function $K_\alpha(t, s)$ as a function of t, is a member of $E(R^n)$ where

$$K_\alpha(t,s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}} \, e^{i\frac{s^2}{2}\cot\phi} . e^{i\frac{t^2}{2}\cot\phi} \, \frac{1}{2}\left[(1 - ie^{i\phi}) cas(\csc\phi.st) + (1 + ie^{i\phi}) cas(-\csc\phi.st) \right],$$

and $\varphi = \dfrac{\alpha\pi}{2}$.

The generalized fractional Hartley transform of $f(t) \in E'(R^n)$, where $E'(R^n)$ is the dual of the testing function space, can be defined as,

$$H^\alpha \left\{ f(t) \right\}(s) = \left\langle f(t), K_\alpha(t,s) \right\rangle. \qquad (2.2.1)$$

Another simple form of fractional Hartley transform as in Sontakke [6] is

$$H^\alpha \left\{ f(t) \right\}(s) = \sqrt{\frac{1 - i\cot\phi}{2\pi}} e^{i\frac{s^2}{2}\cot\phi} \int_{-\infty}^{\infty} e^{i\frac{t^2}{2}\cot\phi} \left[\cos(\csc\phi.st) - ie^{i\phi} \sin(\csc\phi.st) \right] f(t)\, dt \qquad (2.2.2)$$

3. FRACTIONAL HARTLEY TRANSFORM AS AN OPERATOR

In this section we have proved some operation transform formulae for fractional Hartley transform.

3.1. FRACTIONAL HARTLEY TRANSFORM OF PRODUCT OF FUNCTIONS BY USING EIGEN FUNCTION

We wish to obtain the fractional Hartley transform of $x^m(f(t))$ where $f(t)$ stands for any function belonging to the Lebesgue class L^2 in the interval $(-\infty, \infty)$, it is instructive, however to obtain this result directly from the defining eigen value $H^{\alpha} e^{-\frac{t^2}{2}} H_n(t) = e^{-in\alpha} e^{-\frac{t^2}{2}} H_n(t)$ (if n is even).

Using the recurrence relation

$$H_{n+1}(t) + 2nH_{n-1}(t) - 2tH_n(t) = 0 .$$

$$H_n(t) = \frac{1}{2t} H_{n+1}(t) + \frac{n}{t} H_{n-1}(t)$$

and

$$H_{n+1}(t) = 2tH_n(t) - 2nH_{n-1}(t) .$$

$$\therefore H^{\alpha}\left\{ te^{-\frac{t^2}{2}} H_n(t) \right\} = H^{\alpha}\left(te^{-\frac{t^2}{2}} \left[\frac{1}{2t} H_{n+1}(t) + \frac{n}{t} H_{n-1}(t) \right] \right)$$

$$H^{\alpha}\left\{ te^{-\frac{t^2}{2}} H_n(t) \right\} = te^{-i(n+1)\alpha} e^{-\frac{t^2}{2}} H_n(t) + ne^{-\frac{t^2}{2}} \left(e^{-i(n-1)\alpha} - e^{-i(n+1)\alpha} \right) H_{n-1}(t) . \qquad (3.1.1)$$

On the other hand, using $H_n^{'}(x) = 2nH_{n-1}(x)$, we obtain

$$\frac{d}{dt} H^{\alpha}\left\{ e^{-\frac{t^2}{2}} H_n(t) \right\} = \frac{d}{dt}\left(e^{-in\alpha} . e^{\frac{-t^2}{2}} H_n(t) \right)$$

$$= -te^{-in\alpha} e^{\frac{-t^2}{2}} H_n(t) + 2ne^{-in\alpha} e^{\frac{-t^2}{2}} H_{n-1}(t) . \qquad (3.1.2)$$

Eliminating $ne^{-in\alpha} e^{\frac{-t^2}{2}} H_{n-1}(t)$ between equation (3.1.1) and (3.1.2),

$$H^{\alpha}\left\{ te^{-\frac{t^2}{2}} H_n(t) \right\} = te^{-i(n+1)\alpha} e^{-\frac{t^2}{2}} H_n(t) + 2ine^{-\frac{t^2}{2}} e^{-in\alpha} \sin\alpha H_{n-1}(t) . \qquad (3.1.3)$$

and from (3.1.2)

$$2ne^{-in\alpha} e^{\frac{-t^2}{2}} H_{n-1}(t) = \frac{d}{dt} H^{\alpha}\left\{ e^{-\frac{t^2}{2}} H_n(t) \right\} + te^{-in\alpha} e^{\frac{-t^2}{2}} H_n(t)$$

put this in (3.1.3), we get

$$H^{\alpha}\left\{ te^{-\frac{t^2}{2}} H_n(t) \right\} = \left(t\cos\alpha + i\sin\alpha \frac{d}{dt} \right) H^{\alpha}\left\{ e^{-\frac{t^2}{2}} H_n(t) \right\} .$$

Takking $e^{-\frac{t^2}{2}} H_n(t) = f(t) = f$ as in [2] then

$$H^{\alpha}\{tf(t)\} = \left(t\cos\alpha + i\sin\alpha \frac{d}{dt}\right) H^{\alpha}\{f(t)\} \quad (3.1.4)$$

The operator form of this equation is

$$H^{\alpha}\{t\} = \left(t\cos\alpha + i\sin\alpha \frac{d}{dt}\right) H^{\alpha}$$

Repeated use of equation (3.1.4) yield

$$H^{\alpha}\{t^m\} = \left(t\cos\alpha + i\sin\alpha \frac{d}{dt}\right)^m H^{\alpha} \quad (3.1.5)$$

From this equation we immediately find that

$$H^{\alpha}\{t^2 f\} = \left(t\cos\alpha + i\sin\alpha \frac{d}{dt}\right)^2 H^{\alpha}\{f\}$$

$$= \frac{1}{2}\sin 2\alpha (i + t^2 \cos\alpha) H^{\alpha}(f) + it\sin 2\alpha \frac{d}{dt} H^{\alpha}(f) - \sin^2\alpha \frac{d^2}{dt^2} H^{\alpha}(f) \quad (3.1.6)$$

Consider now a function *g(t)*, assumed expandable in a Taylor series $g(t) = \sum b_m t^m$, using equation (3.1.5) we find the more general operator equation

$$H^{\alpha}\{g(t)\} = g\left(t\cos\alpha + i\sin\alpha \frac{d}{dt}\right) H^{\alpha} \quad (3.1.7)$$

Apply equation (3.1.7) to a function *f(t)*, we obtain

$$H^{\alpha}\{gf\} = g\left(t\cos\alpha + i\sin\alpha \frac{d}{dt}\right) H^{\alpha}(f) \quad (3.1.8)$$

Incidentally interchanging the roles of *f* and *g*, we also find

$$H^{\alpha}\{gf\} = f\left(t\cos\alpha + i\sin\alpha \frac{d}{dt}\right) H^{\alpha}(g) \quad (3.1.9)$$

3.2. DIFFERENTIATION RULE

If $f(t) \in E'(R^n)$ and $H^{\alpha}\{f(t)\}(s)$ is fractional Hartley transform of *f(t)* then

$$H^{\alpha}\left\{\frac{d}{dt}f(t)\right\}(s)=i\cot\phi\left[sH^{\alpha}\{f(t)\}(s)-H^{\alpha}\{t(t)\}(s)\right]-s\,H^{\alpha}\{f(-t)\}(s)$$
.

Proof: Since fractional Hartley transform of *f(t)* is

$$H^{\alpha}\{f(t)\}(s)=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\right]f(t)dt$$
.
$$H^{\alpha}\left\{\frac{d}{dt}f(t)\right\}(s)=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\right]f'(t)dt$$
.

Now integrating by parts

$$H^{\alpha}\left\{f'(t)\right\}(s)$$
$$=\sqrt{\frac{1-i\cot\varphi}{2\pi}}e^{i\frac{s^2}{2}\cot\varphi}\left\{\left[e^{i\frac{t^2}{2}\cot\varphi}\cos(\csc\varphi.st)-ie^{i\varphi}\sin(\csc\varphi.st)f(t)\right]_{-\infty}^{\infty}\right.$$
$$-\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\varphi}\left[-\sin(\csc\varphi.st).(\csc\varphi.s)-ie^{i\varphi}\cos(\csc\varphi.st).(\csc\varphi.s)\right]$$
$$\left.+\left[\cos(\csc\varphi.st)-ie^{i\varphi}\sin(\csc\varphi.st)\right]e^{i\frac{t^2}{2}\cot\varphi}\frac{i}{2}.\cot\varphi.2t\right\}f(t)dt$$
$$=\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\left\{\csc\phi.s\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\phi}\left[\sin(\csc\phi.st)+ie^{i\phi}\cos(\csc\phi.st)\right]f(t)dt\right.$$
$$-i\cot\phi.\sqrt{\frac{1-i\cot\phi}{2\pi}}e^{i\frac{s^2}{2}\cot\phi}\int_{-\infty}^{\infty}e^{i\frac{t^2}{2}\cot\phi}\left[\cos(\csc\phi.st)-ie^{i\phi}\sin(\csc\phi.st)\right]t\,f(t)dt$$
$$H^{\alpha}\left\{f'(t)\right\}(s)=i\cot\phi.\left[sH^{\alpha}\{f(t)\}(s)-H^{\alpha}\{t\,f(t)\}(s)\right]-sH^{\alpha}\{f(-t)\}(s)\,. \quad (3.2.1)$$

We have to obtain the transform of the derivative of a function, we use $H^{\alpha}\left\{\frac{d}{dt}f(t)\right\}(s)$, assuming that $f(t)\to 0$ when $x\to\pm\infty$.

Using equation (3.1.4) in equation (3.2.1)

$$H^{\alpha}\left\{\frac{d}{dt}f(t)\right\}(s)=i\cot\phi\,sH^{\alpha}\{f(t)\}(s)-sH^{\alpha}\{f(-t)\}(s)$$
$$-i\cot\phi\left(t\cos\phi+i\sin\phi\frac{d}{dt}\right)H^{\alpha}\{f(t)\}(s).$$

If we consider function is even that is $f(-t)=f(t)$

$$=(i\cot\phi\,s-s)H^{\alpha}\{f(t)\}(s)-i\cot\phi\left(t\cos\phi+i\sin\phi\frac{d}{dt}\right)H^{\alpha}\{f(t)\}(s)$$
$$H^{\alpha}\left\{\frac{d}{dt}f(t)\right\}(s)=\left(i\cot\phi(s-t\cos\phi)-s+\cos\phi\frac{d}{dt}\right)H^{\alpha}\{f(t)\}(s)\,. \quad (3.2.2)$$

The operator form of equation (3.2.2) is

$$H^{\alpha}\left(\frac{d}{dt}\right)=\left(i\cot\phi(s-t\cos\phi)-s+\cos\phi\frac{d}{dt}\right)H^{\alpha}. \tag{3.2.3}$$

And we can immediately extend it to higher derivatives

$$H^{\alpha}\left(\frac{d^m}{dt^m}\right)=\left(i\cot\phi(s-t\cos\phi)-s+\cos\phi\frac{d}{dt}\right)^m H^{\alpha}. \tag{3.2.4}$$

Applying equation (3.2.4) we find

$$H^{\alpha}\left(\frac{d^2 f(t)}{dt^2}\right)(s)=\left(i\cot\phi(s-t\cos\phi)-s+\cos\phi\frac{d}{dt}\right)^2 H^{\alpha}\{f(t)\}(s)$$

$$=\left(-\cot^2\varphi(s-t\cos\varphi)^2-2is\cot\varphi(s-t\cos\varphi)+s^2-i\cos^2\varphi.\cot\varphi\right)H^{\alpha}\{f(t)\}(s)$$

$$+\left(i\cos\varphi\cot\varphi(s-t\cos\varphi)-2s\cos\varphi+i\cos\varphi\cot\varphi(s-t\cos\varphi)\right)\frac{d}{dt}H^{\alpha}\{f(t)\}(s)$$

$$+\cos^2\varphi\frac{d^2}{dt^2}H^{\alpha}\{f(t)\}(s). \tag{3.2.5}$$

Again if g is function expandable in a Taylor series we have

$$H^{\alpha}g\left(\frac{d}{dt}\right)f=g\left(i\cot\phi(s-t\cos\phi)-s+\cos\phi\frac{d}{dt}\right)H^{\alpha}\{f\} \tag{3.2.6}$$

Now we consider function is odd that is $f(-t)=-f(t)$ then

$$H^{\alpha}\left\{\frac{d}{dt}f(t)\right\}(s)=\left(i\cot\phi(s-t\cos\phi)+s+\cos\phi\frac{d}{dt}\right)H^{\alpha}\{f(t)\}(s).$$

The operator form is

$$H^{\alpha}\left(\frac{d}{dt}\right)=\left(i\cot\phi(s-t\cos\phi)+s+\cos\phi\frac{d}{dt}\right)H^{\alpha}.$$

And can immediately extend it to higher derivatives

$$H^{\alpha}\left(\frac{d^m}{dt^m}\right)=\left(i\cot\phi(s-t\cos\phi)+s+\cos\phi\frac{d}{dt}\right)^m H^{\alpha}$$

and for second order derivative

$$H^{\alpha}\left(\frac{d^2 f(t)}{dt^2}\right)(s)=\left(i\cot\phi(s-t\cos\phi)+s+\cos\phi\frac{d}{dt}\right)^2 H^{\alpha}\{f(t)\}(s)$$

$$= \left(-\cot^2\phi(s - t\cos\phi)^2 + 2is\cot\phi(s - t\cos\phi) + s^2 - i\cos^2\phi.\cot\phi\right)H^\alpha\{f(t)\}(s)$$
$$+ \left(i\cos\phi\cot\phi(s - t\cos\phi) + 2s\cos\phi + i\cos\phi\cot\phi(s - t\cos\phi)\right)\frac{d}{dt}H^\alpha\{f(t)\}(s)$$
$$+ \cos^2\phi\frac{d^2}{dt^2}H^\alpha\{f(t)\}(s).$$

Again if g is function expandable in a Taylor series we have

$$H^\alpha g\left(\frac{d}{dt}\right)f = g\left(i\cot\phi(s - t\cos\phi) + s + \cos\phi\frac{d}{dt}\right)H^\alpha\{f\}. \tag{3.2.7}$$

3.3. THE MIXED PRODUCT RULE

A rule for the product $t\left(\frac{df}{dt}\right)$ can be obtained most easily by using equation (3.1.4) and (3.2.2)

$$H^\alpha\left\{t\frac{df}{dt}\right\} = \left(t\cos\phi + i\sin\phi\frac{d}{dt}\right)H^\alpha\left\{\frac{df}{dt}\right\},$$

suppose $\frac{df}{dt} = f(t)$ then

$$= \left(t\cos\phi + i\sin\phi\frac{d}{dt}\right)\left(i\cot\phi(s - t\cos\phi) - s + \cos\phi\frac{d}{dt}\right)H^\alpha\{f(t)\}(s)$$
$$= \left(t\cos\phi + i\sin\phi\frac{d}{dt}\right)\begin{pmatrix} i\cot\phi(s - t\cos\phi)H^\alpha\{f(t)\}(s) - sH^\alpha\{f(t)\}(s) \\ + \cos\phi\frac{d}{dt}H^\alpha\{f(t)\}(s) \end{pmatrix}$$
$$= \left(it\cos\varphi\cot\varphi(s - t\cos\varphi) - st\cos\varphi + \cos\varphi\cot\varphi\right)H^\alpha\{f(t)\}(s) +$$
$$\left(t\cos^2\varphi - \cos\varphi(s - t\cot\varphi) - is\sin\varphi\right)\frac{d}{dt}H^\alpha\{f(t)\}(s) - i\sin\varphi\cos\varphi\frac{d^2}{dt^2}H^\alpha\{f(t)\}(s) \tag{3.3.1}$$

3.4. THE INTEGRATION RULE

To form fractional Hartley transform of integration of function, we use equation (3.2.2),

$$H^\alpha\left\{\frac{dg(t)}{dt}\right\} = H^\alpha\left\{\frac{dg}{dt}\right\} = \left(i\cot\phi(s - t\cot\phi) - s\right)H^\alpha\{g(t)\} + \cos\phi\frac{d}{dt}H^\alpha\{g(t)\}.$$

Letting $\frac{dg(t)}{dt} = f(t) \therefore g(t) = \int f(t)dt$,

$$H^\alpha\{f(t)\} = \left(i\cot\phi(s - t\cot\phi) - s\right)H^\alpha\left\{\int_0^t f(t)dt\right\} + \cos\phi\frac{d}{dt}H^\alpha\left\{\int_0^t f(t)dt\right\}.$$

Hence

$$\frac{d}{dt}\left[H^{\alpha}\left\{\int_0^t f(t)dt\right\}\right]+\left(i\cos ec\phi(s-t\cot\phi)-s\sec\phi\right)H^{\alpha}\left\{\int_0^t f(t)dt\right\}=\sec\phi H^{\alpha}\{f(t)\}. \tag{3.4.1}$$

This is linear differential equation, therefore solution is

$$H^{\alpha}\left\{\int_0^t f(t)dt\right\}\exp\left((is\cos ec\phi-\sec\phi)t-i\frac{t^2}{2}\cos ec\phi\cot\phi\right)$$
$$=\int\sec\phi H^{\alpha}\{f(t)\}\exp\left((is\cos ec\phi-\sec\phi)t-i\frac{t^2}{2}\cos ec\phi\cot\phi\right)dt+c.$$

That is

$$H^{\alpha}\left\{\int_0^t f(t)dt\right\}=\sec\phi\left[\exp\left((is\cos ec\phi-\sec\phi)t-i\frac{t^2}{2}\cos ec\phi\cot\phi\right)\right]^{-1}$$
$$.\int H^{\alpha}\{f(t)\}\exp\left((is\cos ec\phi-\sec\phi)t-i\frac{t^2}{2}\cos ec\phi\cot\phi\right)dt+c.$$

4. SOLUTION OF TIME INDEPENDENT SCHRÖDINGER EQUATION FOR THE QUANTUM HARMONIC OSCILLATOR

We now proceed to apply the rules of the generalized operational calculus to the solution of the time- independent Schrodinger equation for the harmonic oscillator.

$$-\frac{h^2}{2m}\frac{d^2\psi}{dx^2}+\frac{1}{2}kx^2\psi=F_{\psi}, \tag{4.1}$$

where h is Plank's constant k is the spring constant for the oscillator of mass m and energy E. This equation can be put in a reduced form by letting $t=\sqrt[4]{4mk/h^2x}$ and $\frac{E}{h}\sqrt{\frac{k}{m}}=\lambda$ $\gamma=\frac{1}{2}$

$$\frac{d^2\psi}{dt^2}+(\lambda-\gamma^2t^2)\psi=0 . \tag{4.2}$$

We know that the fractional Hartley transform of equation (4.2)

$$H^{\alpha}\left\{\frac{d^2\psi}{dt^2}\right\}+\lambda H^{\alpha}(\psi)-\gamma^2H^{\alpha}(t^2\psi)=0 \tag{4.3}$$

Using the rules (3.1.6) and (3.2.5) and letting $H^{\alpha}(\psi)=G$ and ψ even function, we find that G satisfies the second order differential equation.

$$\left(-\cot^2\phi(s-t\cos\phi)^2-2is\cot\phi(s-t\cos\phi)+s^2-i\cos^2\phi\cot\phi\right)H^\alpha\{\psi(t)\}$$
$$+\left(i\cos\phi\cot\phi(s-t\cos\phi)-2s\cos\phi+i\cos\phi\cot\phi(s-t\cos\phi)\right)\frac{d}{dt}H^\alpha\{\psi(t)\}$$
$$+\cos^2\phi\frac{d^2}{dt^2}H^\alpha\{\psi(t)\}+\lambda H^\alpha\{\psi(t)\}$$
$$-\gamma^2\left(\frac{1}{2}\sin 2\phi(i+t^2\cot\phi)H^\alpha\{\psi(t)\}+it\sin 2\phi\frac{d}{dt}H^\alpha\{\psi(t)\}-\sin^2\phi\frac{d^2}{dt^2}H^\alpha\{\psi(t)\}\right)=0$$

$$\left(\cos^2\phi+\gamma^2\sin^2\phi\right)\frac{d^2}{dt^2}H^\alpha\{\psi(t)\}+\begin{pmatrix}2i\cos\phi\cot\phi(s-t\cos\phi)\\-2s\cos\phi-i\gamma^2t\sin 2\phi\end{pmatrix}\frac{d}{dt}H^\alpha\{\psi(t)\}$$
$$+\begin{pmatrix}-\cot^2\phi(s-t\cos\phi)^2-2is\cot\phi(s-t\cos\phi)\\+s^2-i\cot^2\phi-\frac{\gamma^2}{2}\sin 2\phi(i+t^2\cot\phi)+\lambda\end{pmatrix}H^\alpha\{\psi(t)\}=0$$

$$\left(\cos^2\phi+\gamma^2\sin^2\phi\right)G''+\left(2i\cos\phi\cot\phi(s-t\cos\phi)-2s\cos\phi-i\gamma^2t\sin 2\phi\right)G'+$$
$$\left(-\cot^2\phi(s-t\cos\phi)^2-2is\cot\phi(s-t\cos\phi)+s^2-i\cot^2\phi-\frac{\gamma^2}{2}\sin 2\phi(i+t^2\cot\phi)+\lambda\right)G=0 \tag{4.4}$$

we now reduce this equation to first order by setting $\cos^2\varphi+\gamma^2\sin^2\varphi=0$. Thus the reduction to first order can simply be accomplished by using a fractional transform with angle φ such that $\cos^2\varphi=-\gamma^2\sin^2\varphi$, $\cot^2\varphi=-\gamma^2$,

$$\cot\varphi=\pm i\gamma\,. \tag{4.5}$$

We shall write this equation $\cot\varphi=i\in\gamma$ where $\in=\pm$, and the proper sign will be determined by examining the behavior of the integrand in the final result.

For the harmonic oscillator $\gamma=\frac{1}{2}$ and the angle φ which satisfy equation (4.5) is complex corresponding to a fractional Hartley transform of complex order. Using equation (4.4), we obtain

$$\left(2i\cos\phi\, i\in\gamma(s-t\cos\phi)-2s\cos\phi-i\gamma^2t\sin 2\phi\right)G'+$$
$$\left(-(i\in\gamma)^2(s-t\cos\phi)^2-2isi\in\gamma(s-t\cos\phi)+s^2-i(i\in\gamma)^2-\frac{\gamma^2}{2}\sin 2\phi(i+t^2i\in\gamma)+\lambda\right)G=0$$
$$\left(-2\in\gamma\cos\phi(s-t\cos\phi)-2s\cos\phi-i\gamma^2t\sin 2\phi\right)G'+$$
$$\left(\gamma^2(s-t\cos\phi)^2+2s\in\gamma(s-t\cos\phi)+s^2+i\gamma^2-\frac{\gamma^2}{2}\sin 2\phi(i+t^2i\in\gamma)+\lambda\right)G=0$$
$$G'+\frac{\left(\gamma^2(s-t\cos\phi)^2+2s\in\gamma(s-t\cos\phi)+s^2+i\gamma^2-\frac{\gamma^2}{2}\sin 2\phi(i+t^2i\in\gamma)+\lambda\right)}{\left(-2\in\gamma\cos\phi(s-t\cos\phi)-2s\cos\phi-i\gamma^2t\sin 2\phi\right)}G=0$$
$$G'+f(t)G=0 \tag{4.6}$$

where

$$f(t)=\frac{\left(\gamma^2(s-t\cos\phi)^2+2s\in\gamma(s-t\cos\phi)+s^2+i\gamma^2-\frac{\gamma^2}{2}\sin 2\phi(i+t^2 i\in\gamma)+\lambda\right)}{\left(-2\in\gamma\cos\phi(s-t\cos\phi)-2s\cos\phi-i\gamma^2 t\sin 2\phi\right)}$$

$$G^{'}=-f(t)G$$

That is

$$\int\frac{G^{'}}{G}dt=-\int f(t)\,dt$$

$$\log G=-\int f(t)\,dt\,.$$

Therefore

$$G=\exp(-\int f(t)\,dt)\,.$$

$$H^{\alpha}(\psi)=\exp(-\int f(t)\,dt)\,.$$

This is required solution of Time Independent Schrödinger Equation for the Quantum Harmonic Oscillator.

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