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STUDIES AND APPLICATIONS OF ABSOLUTE STABILITY OF THE NONLINEAR DYNAMICAL SYSTEMS

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Abstract. *In this paper there are presented the methods of study for the automatic regulation of the absolute stability of the nonlinear dynamical systems. There are specified two methods for the absolute stability with the criteria and the type of application: a) the Lurie method with the effective determination of the Liapunov function; b) the frequencies method of the Romanian researcher V. M. Popov using the transfer function in the critical cases. The nonlinear dynamical systems reefers to the special classes include the linear and nonlinear blocks, due to the perturbations composed with the inverse actions of response of the automatic regulators components to obtain the stable absolute regime. The mathematical methodology is analytical numerically and the application used by the two methods is from the metal cutting tools machine and in the absolute stability of the rate of aircraft (airplane) equipped with autopilot.*

1. INTRODUCTION

The automatic regulation for the stability of dynamical systems occupies a fundamental position in science and technique, following the optimization of the technological process of the cutting tools, of the robots, of the movement vehicles regime or of some machines components, of energetic radioactive regimes, chemical, electromagnetic, thermal, hydro-aerodynamic regimes, etc.

The studies and the technical achievements are complex by mathematical models for closed circuits with input - output, following for the automatic regulation the integration of some mechanisms and devices with inverse reaction of response for the control and the fast and efficient elimination of the perturbations which can appears along these processes or dynamical regimes. Generally these dynamical regimes are nonlinear and it was necessarily some contributions and special achievements for automatic regulation, generating the automatic regulation of absolute stability (a.r.a.s.) for these classes of nonlinearities.

We highlight two special methods (a.r.a.s.): • Liapunov's function method discovered by A.I. Lurie [13, 15, 20] and developed into a series of studies by M.A Aizerman, V.A. Iakubovici, F.R. Gantmaher, R.E. Kalman, D.R. Merkin [14] and others [1, 17].

Frequency method developed by researcher VM Popov [18] generalizing the criterion of Nyquist, then developed in many studies [1, 2, 15].

We note the contributions of Romanian researchers recognized by the works and monographs on the stability and optimal control theory: C. Corduneanu, A. Halanay, V. Barbu, Th. Morozan, G. Dinca, M. Megan, Vl. Rasvan, V. Ionescu, M.E. Popescu, S.

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Chiriacescu, A. Georgescu and also who studied directly on (a.r.a.s.): I. Dumitrache [4] D. Popescu [16], C. Belea [2], V. Rasvan [19], S. Chiriacescu [3] and other recent works [6-12].

The research has shown that both methods are equivalent, and studies can be qualitatively or numerically. In this paper we presented the actual making methods in cases of singularity studies across applications.

2. (A.R.A.S.) USING THE LIAPUNOV'S FUNCTION METHOD

In this part we'll present the Lurie's ideas and the effective method for found the Liapunov's function [13, 14, 2, 19]. Generally, the systems of automatic regulation are composed from the controlled processor system, and sensory elements of measurement, acquisition board, and the mechanism feedback controller. The regulator will mean all the sensors and the acquisition board, but the controller is included feedback mechanism. Parameters characterizing the object control system to control work mode are measured by sensors, and their records with the sensor response mechanism ζ is transmitted acquisition board. This processes the command σ, which is mechanically transmitted to the controller which, on its turn, distributes the object state and interact simultaneously adjusting the response mechanism. We highlight the dynamic system equations. We note by $x_1, x_2, \ldots, x_n$ the state parameters of the regime's subject which it must controlled, the coordinates and the sensorial speeds. We rename that the variation of these parameters if the open circuit (excluding the controller) system described by linear differential equations with constant coefficients: $\dot{x}_k = \sum_{j=1}^{n} a_{kj} x_j, k = 1, \ldots, n$. If the system is with closed loop then on the variables $x_1, x_2, \ldots, x_n$ will influence the regulation body, and we note by ξ its state. In this case for the autonomous closed system we have the equations:

$$\dot{x}_k = \sum_{j=1}^{n} a_{kj} x_j + b_k \xi, k = 1, \ldots, n \tag{1}$$

We'll consider that the mechanism or inverse reaction is determine on the output ζ with the rigidity connection on the input ξ :

$$\zeta = k\xi \tag{2}$$

The acquisition board collects the signals and transmits the input sensors in order to obtain the embedded system:

$$\sigma = \sum_{j=1}^{n} c_j x_j - r\xi \tag{3}$$

where c_j, r are transfer numbers, r is the transfer coefficient of the inverse rigid connection, $r > 0$ (the regulator characteristics) [13,14,15]. The connection between the output function σ (linear) of the controller and the nonlinear input φ in the case of automatic regulation is express by the relation:

$$\dot{\xi} = \varphi(\sigma) \tag{4}$$

The characteristic function of the controller $\varphi(\sigma)$, $\sigma \in (-\infty, +\infty)$ is continuous and verify the conditions [14, 6, 7]:

$$\begin{aligned} &a) \quad \varphi(0) = 0 \\ &b) \quad \sigma \cdot \varphi(\sigma) > 0, \quad \forall \sigma \neq 0 \\ &c) \quad \int_0^{\pm\infty} \varphi(\sigma) d\sigma = \infty \end{aligned} \tag{5}$$

Observe that $\varphi = \varphi(\sigma)$ is ascending in the quarters I, III where is graphically. The functions $\varphi(\sigma)$ are named admissible, and is verified the sector condition:

$$0 < \frac{\varphi(\sigma)}{\sigma} < k \tag{6}$$

where k is the amplification coefficient.

Example 1.

- $\varphi(\sigma) = sgn(\sigma) \cdot \ln(\sigma^2 + 1), k > 1$
- $\varphi(\sigma) = a(e^{\sigma} - 1), k \leq a$

The equations (1), (3), (4) model the perturbed system with the zeros $x(0,0,\ldots,0)$, $\xi = 0$.

Using the nonsingular square matrix $A = \|a_{kj}\|$ of degree $n > 1$, $B = \begin{pmatrix} b_1 \\ \cdots \\ b_n \end{pmatrix}$, $C = (c_1 \quad \cdots \quad c_n)$, C' the transpose matrix of C, this system can be:

$$\dot{X} = AX + B\xi, \quad \dot{\xi} = \varphi(\sigma),$$
$$\sigma = C'X - r\xi, \quad X = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix} \tag{7}$$

Observation. It is known that for the linear system $\dot{X} = AX$, the second method of Liapunov for the null solution stability consists in determine a Liapunov function $V = V(x)$ fulfilled the regularity conditions associated of this system [1,20]. A simple technique is to search V like square form positive defined $V = X'PX$ and $\dot{V} = X'(A'P + PA)X$ associated of the autonomous system where $V(0) = 0, \dot{V}(0) = 0$. For the simple or asymptotic stability in the vicinity of the null solution must have negative sign (or negative defined). It must:

$$A'P + PA = -Q \tag{*}$$

Where the matrix $P, Q \in \mathbf{R}_{n\times n}$ Q are symmetrically and positives. So, practically it is choose Q randomize fixed and is determined the matrix P from the equation (*) with A nonsingular.

Bringing the system (7) to the canonical form and determine the Liapunov function:

Suppose that A with $\det A = \Delta_0 \neq 0$ is Hurwitz, that mean the characteristic polynomial $P(\lambda)$ has simple roots with $Re(\lambda_k) < 0, k = 1,\ldots,n$

$$P(\lambda) = (-1)^n \det(A - \lambda E) = 0 \tag{8}$$

The system (7) is bring to the canonical form if the matrix A is bring to the Jordan form $J = diagA = \begin{pmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{pmatrix}$. It is determine a non degenerate matrix $T = (t_{kj})$ for the diagonalization of matrix A with the relation:

$$T^{-1}AT = J, \quad AT = TJ, \quad \det T \neq 0 \tag{9}$$

We make the linear transform:

$$X = TY, Y = \begin{pmatrix} y_1 \\ \cdots \\ y_n \end{pmatrix} \tag{10}$$

Obtaining from (7):

$$T\dot{Y} = ATY + B\xi, \quad \dot{\xi} = \varphi(\sigma), \sigma = C'TY - r\xi$$

that mean:

$$\dot{Y} = JY + B_1\xi, \quad \dot{\xi} = \varphi(\sigma),$$
$$\sigma = C_1'Y - r\xi, B_1 = T^{-1}B, C_1' = C'T \tag{11}$$

Reducing the system (1) with the linear transform:

$$Z = JY + B_1\xi, \sigma = C_1'Y - r\xi, Z = \begin{pmatrix} z_1 \\ \cdots \\ z_n \end{pmatrix} \tag{12}$$

$$\begin{cases} \dot{Z} = JZ + B_1\varphi(\sigma) \\ \dot{\sigma} = C_1'Z - r\varphi(\sigma) \end{cases} \tag{13}$$

The disturbed system (13) with the equilibrium solution ($z_k = 0, \sigma = 0)$ will be equivalent with the system (7) with the equilibrium solution $(x_k = 0, \xi = 0)$ and the transform (12) will be non degenerate if the determinant of the system (13) is non null.

$$\Delta = \begin{vmatrix} J & B_1 \\ C_1' & -r \end{vmatrix} \neq 0, r + C_1'J^{-1}B_1 \neq 0 \tag{14}$$

Retuning to $J^{-1} = T^{-1}AB, B_1 = T^{-1}B, C_1' = C'T$ transforms we obtain from (14) the final condition:

$$r + C'A^{-1}B \neq 0 \tag{15}$$

The Lurie's problem consists in calculus the asymptotic stability conditions of the (7) equivalent with (13) with the null solution respectively $(x_k = 0, \xi = 0)$, $(z_k = 0, \sigma = 0)$ for the initial perturbations and for any admissible functions $\varphi(\sigma)$ defined in (5), (6). This type of stability where the systems (7), (13) have a linear part which is the A and a non linear part which is $\varphi(\sigma)$ is named the absolute stability (a.s), [1,16] It is observe that if $\varphi(\sigma)$ is linear, than the systems are linearized being asymptotic stable. The simplicity of system (13) entails immediate techniques for determining the Liapunov function $V = V(z_1, \ldots, z_n, \sigma)$ attach to the system (13). The function $V(z, \sigma)$ of class C^1 is Liapunov from the system (13) if $V(z = 0, \sigma = 0) = 0$ and is positive defined $V(z, \sigma) > 0$ radial unlimited to ∞ , with the absolute derivative $\dot{V} = \frac{dV}{dt}$ $\dot{V}(0,0) = 0$ and $\dot{V}$ negative defined $\frac{dV}{dt} < 0$ for $(z \neq 0, \sigma \neq 0)$ in vicinity of the equilibrium point for have than absolute stability. Here, for the case of automatic regulation we choose $V, \dot{V}$ have the special form which verify these conditions. So we search the function $V = V(z,\sigma)$ compose by a square form z_k corresponding to the linear block A and an integral term corresponding to the non linear part.

$$V(z,\sigma) = Z'PZ + \int_0^\sigma \varphi(\sigma)d\sigma = V_1(z,\sigma) + \int_0^\sigma \varphi(\sigma)d\sigma \tag{16}$$

From theory [1,4] $Z'PZ$ is the square form defined strictly positive if the matrix P is symmetric $(P = P')$ and we have $A'P + PA = -Q$ where Q is symmetric and positive (with the eigenvalues positive). The integral term from (16) is strictly positive from the conditions (5) with $\sigma \neq 0$ and $V(z = 0, \sigma = 0) = 0$. Next are verify the regularity conditions with $\dot{V}$ attach to (13) and with (15) will obtain the conditions for parameters c_k, r to obtain (a.r.a.s.). From (16) using (13) and:

$$Q = Q', P = P', B_1'PZ + Z'PB_1 = B_1'PZ + (PB_1)'Z = 2(PB_1)'Z$$

for:

$$\frac{dV(z,\sigma)}{dt} = Z'(J'P + PJ)Z - r\varphi^2(\sigma) + \varphi(\sigma)(B_1'PZ + Z'PB_1) + \varphi(\sigma)C_1$$

We obtain:

$$\frac{dV}{dt} = -Z'QZ - r\varphi^2(\sigma) + 2\varphi(\sigma)\left(PB_1 + \frac{1}{2}C_1\right)Z; \dot{V}(z = 0, \sigma = 0) = 0 \tag{17}$$

It can be see the connection from the matrix components $P(p_{ij}), Q(q_{ij})$ from $\lambda_i + \lambda_j \neq 0, i, j = 1, \ldots, n, P = P', J = \text{diag}A$ than from $Q = Q'$ we have $q_{ij} = -\left(\lambda_i p_{ij} + \lambda_j p_{ij}\right)$ that mean:

$$p_{ij} = -\frac{q_{ij}}{\lambda_i + \lambda_j} \tag{18}$$

Observation 1. The matrix A is stable with $\lambda_i + \lambda_j \neq 0$ if Q is a square form positive defined.

Example 2. If choose $Q = E$ the unit matrix and P obtain from (18) than the below observation is valid. Because $\dot{V} < 0$ we prove that $(-\dot{V})$ is positive defined. Apply in (17) the Silvester criterion demanding that all diagonal minors of (17) to be positive. Because Q is positive like square form, than the first n inequalities are verify; it rest the last inequality from (17) after the square form in z and which is:

$$r > \left(PB_1 + \frac{1}{2}C_1\right)' Q^{-1}\left(PB_1 + \frac{1}{2}C_1\right) \tag{19}$$

For

$$Q = E, \sqrt{r} > \left\| PB_1 + \frac{1}{2}C_1 \right\|.$$

If the regulator parameters verify the conditions (15), (19) there are sufficient conditions for the asymptotic stability of the system (1), (3), (4) for the solution $(x=0,\xi=0)$. [13, 19, 11].

Remark 1. A choice technique of the square form $V_1(z)$ for p_{ij} according Lurie is:

$$V_1(z)=\varepsilon\sum_{k=1}^{s} z_{2k-1}z_{2k}+\frac{\varepsilon}{2}\sum_{k=1}^{n-2s} z_{2s+k}^2-\sum_{k=1}^{n}\sum_{j=1}^{n}\frac{a_k z_k a_j z_j}{\lambda_k+\lambda_j},\ \varepsilon>0$$

where $a_1,a_2,\ldots,a_{2s}$ are complex conjugated, $a_{2s+1},\ldots,a_n$ are real corresponding to roots λ_k determining the coefficients a_k .

Remark 2. The two transforms for the diagonal system (1), (3), (4) to obtain (13) can be replacing directly with the transform [15]:

$$x_k=-\sum_{i=1}^{n}\frac{N_k(\lambda_i)}{D'(\lambda_i)}z_i \tag{20}$$

where from (7) $P(\lambda)=(-1)^n D(\lambda), N_k(\lambda)=\sum_{i=1}^{n} b_i D_{ik}(\lambda)$, D_{ik} are the corresponding algebraic complements of (i,k) from $D(\lambda)=A-\lambda E$. In this case the simplified system analogous (13):

$$\dot z_k=\lambda_k z_k+\varphi(\sigma), \dot\sigma=\sum_{i=1}^{n} f_i z_i-r\varphi(\sigma), k=1,...,n \tag{21}$$

for which we will build easier $V(z,\varphi)$.

Determining of $V(z,\varphi)$ with a new efficient method for (13) or (21)

Following the form of $V_1(z)$ we choose the function $V(z,\sigma)$ for (21).

$$V(z,\sigma)=\frac{1}{2}\sum_{j=1}^{n} A_j z_j^2+F(\alpha_1 z_1,\alpha_2 z_2,...,\alpha_n z_n)+\int_0^{\sigma}\varphi(\sigma)d\sigma \tag{22}$$

$$F(z_1,z_2,...,z_n)=-\sum_{j.k=1}^{n}\frac{1}{\lambda_j+\lambda_k} z_j z_{k,}\lambda_{k<0} \tag{23}$$

where, $A_j>0,\alpha_j\in\mathbf{R}$ will be determined. From:

$$-\frac{1}{\lambda_j+\lambda_k}=\int_0^{\infty} e^{(\lambda_j+\lambda_k)s}ds>0$$

$$F(z_1,z_2,...,z_n)=\int_0^{\infty}\sum_{j,k} z_j z_k e^{(\lambda_j+\lambda_k)s}ds=\int_0^{\infty}\left(\sum_{j=1}^{\infty} z_j e^{\lambda_j s}\right)^2 ds\geq 0$$

Results that F is nullify just for $F(z_1=0,z_2=0,...,z_n=0)=0$ and $\int_0^{\sigma}\varphi(\sigma)d\sigma>0$.

So, $V(z,\sigma)$ has the positive sign defined and $V(z=0,\sigma=0)=0$. Compute $\frac{dV}{dt}$ associate to the system (21) and it must be $(-\dot V)$ of positive sign defined.

$$-\frac{dV}{dt} = -\sum_{j=1}^{n} A_j \lambda_j z_j^2 - 2\sum_{j,k=1}^{n} \frac{\lambda_j \alpha_j \alpha_k}{\lambda_j + \lambda_k} z_j z_k + r\varphi^2(\sigma) + \sum_{j=1}^{n} z_j \left[A_j + f_j - 2\alpha_j \sum_{k=1}^{n} \frac{\alpha_k}{\lambda_j + \lambda_k} \right] \varphi$$

From $$2\sum_{j,k=1}^{n} \frac{\lambda_j \alpha_j \alpha_k}{\lambda_j + \lambda_k} z_j z_k = \left(\sum_{k=1}^{n} \alpha_k z_k \right)^2, r > 0, \lambda_j > 0$$

We obtain the first three terms positives and must nullifying the coefficient of φ:

$$A_j + f_j - 2\alpha_j \sum_{k=1}^{n} \frac{\alpha_k}{\lambda_j + \lambda_k} = 0, \; j = 1..n \tag{24}$$

In this quadratic algebraic system (24) we can take $A_j = -\frac{1}{\lambda_j}$, and f_j, λ_j known, we determine the coefficients $\alpha_j, j = 1..n$ and other conditions from (19). If in (24) divide with λ_j and summing we obtain

$$\left(\sum_{j=1}^{n} \frac{\alpha_j}{\lambda_j} \right)^2 = -\sum_{j=1}^{n} \frac{A_j + f_j}{\lambda_j} \equiv \Gamma^2, \sum_{j=1}^{n} \frac{\alpha_j}{\lambda_j} = \pm\Gamma \tag{25}$$

So, must have $\sum_{j=1}^{n} \frac{A_j + f_j}{\lambda_j} < 0$, and the solution of the system (24) $(\alpha_1, \alpha_2, ..., \alpha_n)$ is in this hyper-plane (25).

For the case when a root is null $P(0) = 0$ and the others have $Re(\lambda_k) < 0, k = 1,..., n-1$ than the system (13) with $Z = \begin{pmatrix} \tilde{z} \\ z_n \end{pmatrix}$ becomes:

$$\dot{\tilde{z}} = \tilde{J}\tilde{Z} + \tilde{B}_1\varphi, \dot{z}_n = b_0\varphi, \dot{\varepsilon} = \tilde{C}_1'\tilde{Z} + C_0 z_n - r\varphi \tag{26}$$

where for $\tilde{z}$ we have the matrix $\tilde{Z}$ and $\tilde{J}$ of degree $(n-1)$, $\tilde{B}_1, \tilde{C}_1$ row, column matrix $(n-1,1),(1,n-1)$. In this case the Liapunov function search form:

$$V(\tilde{z}, z_1, \sigma) = az_1^2 + \left\{ \tilde{z}' P\tilde{z} + \int_0^{\sigma} \varphi(\sigma) d\sigma \right\} \tag{27}$$

For proofs and recently applications we recommend the bibliography [2, 15, 14, 11, 12].

3. THE FREQUENCY METHOD FOR (A.R.A.S.)

This method obtained by V.M. Popov [18] is applied to the dynamical system with continuous nonlinearity. We present in this section the method with criterions given by Aizerman, Kalman, Jakubovici [19,14]. Let be the dynamical, autonomous, non homogeneous system:

$$\dot{x}_i = \sum_{l=1}^{n} a_{il} x_l + b_i u, i = 1,..., n; \dot{x} = \frac{dx}{dt}$$
$$\sigma = \sum_{l=1}^{n} c_l x_l, u = -\varphi(\sigma) \tag{28}$$

where a_{il}, b_i, c_l are real constants, u is the arbitrary function of input , continuous, nonlinear with $\varphi(\sigma)$ and σ is the output function. Using the Laplace transform, replacing the operator $\frac{d}{dt}$ with s we obtain from (2):

$$sx_i = \sum_{l=1}^{n} a_{il} x_l + b_i u, \sigma = \sum_{l=1}^{n} c_l x_l, i = 1,...,n \tag{29}$$

Eliminating from (21) the characteristic parameters of the regulator is obtained:

$$\sigma = W(s)u, \sigma = W(s)(-\varphi) \tag{30}$$

where $W(s) = \frac{Q_m(s)}{Q_n(s)}$ is the transfer function and $Q(s)$ are polynomials $m < n$ [4, 6, 16]. The transfer function connect σ and φ ; the function φ verify the conditions (5) and the sector condition (6) $0 < \frac{\varphi(\sigma)}{\sigma} < k \le \infty$ - the plot $\varphi = \varphi(\sigma)$ in the plane (σ, φ) will be the sector $0 \le \varphi(\sigma) \le k\sigma$. The sector condition and the nonlinearity of φ determine the system (σ, φ) with closed loop through the impulse function φ . We study the absolute stability of the perturbed system (29) from the null solution $(x = 0, u = 0)$. Because the system is closed and nonlinear we can't applied directly the Nyquist criterion, [4,6,18]. If $\varphi \equiv k\sigma$ then the system is linear and it cab be applied this criterion. It observe that the block $\sum a_{il} x_l$ is linear and $b_i u$ is nonlinear and result that the roots of characteristic polynomial $P(\lambda) = (-1)(A - \lambda E) = 0, P(\lambda_i) = 0,$ the poles of $W(s)$ and k will influence the determination of the absolute stability criteria. From $W(s = j\omega) = U(\omega) + jV(\omega), j = \sqrt{-1}$ we have the hodograph for the axis (U, V) [2, 4, 6, 7, 15]:

$$U = U(\omega), V = V(\omega), 0 \le \omega \le \infty \tag{31}$$

If all poles of $W(s)$ have $Re(s_i) < 0$ then the system is uncritically; if through the poles of $W(s)$ are a part null or on the imaginary axis and the rest have $Re(s_i) < 0$ then the system is in the critical case. We enunciate the criteria for absolute stability of automatic control (a.r.a.s.) by the frequency method.

Criterion 1.*(the uncritically case). Let be the conditions:*

a) The function $\varphi(\sigma)$ verifies (5), (6)

b) All poles of $W(s)$ have $Re(s_i) < 0$

c) If exists there a real number $q \in R$ that $\forall \omega \ge 0$ is satisfied the condition:

$$\frac{1}{k} + Re\left[(1 + j\omega q)W(j\omega)\right] \ge 0 \tag{32}$$

Then the system (20) is automatic regulated and absolute stable for the null solution $(x = 0, u = 0)$.

From (32) is obtained:

$$\frac{1}{k} + U(\omega) - q\omega V(\omega) \ge 0 \tag{33}$$

The criterion (32) geometrically shows that in the plane geometric $U_1 = U, V_1 = \omega V$ exists the line (33) passing through $\left(-\frac{1}{k}, 0\right)$ and the plot of the hodograph is under this line for $\omega \geq 0, k > 0$.

Criterion 2. *(the critical case when there are a simple null pole $s_0 = 0$). Let be satisfied the conditions:*

a) The function φ verify (5), (6).

b) $W(s)$ has a simple null pole, and the others poles s_i have $Re(s_i) < 0$.

c) We have $\rho = \lim_{s \to 0} sW(s) > 0$ and exists $q \in R$ for $\forall \omega \geq 0$ verifying the condition (33) Then for the system (28) for the null solution we have (a.r.a.s.).

Criterion 3. *(the critical case when $s = 0$ is a double pole). Let be the conditions:*

a) The function $\varphi(\sigma)$ verify (5), (6) and the sector condition for $k = \infty$ in the quarters I, III.

b) $W(s)$ has a double pole in s=0 and the others poles has $Re(s_i) < 0$.

c) Is verifying $\rho = \lim_{s \to 0} s^2 W(s) > 0$, $\mu = \lim_{s \to 0} \frac{d}{ds}\left[s^2 W(s)\right] > 0$, $\pi(\omega) = \omega Im W(j\omega) < 0$ for $\forall \omega \geq 0$ then for the system (28) we have (a.r.a.s.) for the null solution.

Observation 2. The shape of these criteria (I, II, III) has an analytical character and their verification is required for construction of hodograph values of the coefficients by numbers. For special cases the recommended monographs are [2,4,15,19].

4. THE STUDY OF THE ABSOLUTE STABILITY OF SOME AIRCRAFT COURSE WITH THE AUTOMATIC PILOT

We'll consider the airplane fly in the vertical plane xOy, the longitudinal axis of the aircraft is parallel with the horizontal axis Ox and the vertical plane is symmetry plane for the aircraft. In the longitudinal fly course (horizontal) can appear some perturbations with angular variations for:

- the pitch(tangage) angle ψ, between the longitudinal axis and Ox
- the speed angle on the trajectory of fly θ, with the axis Ox compared with the considered system $\psi - \theta = \alpha$, represents the attack angle [17].

Considering these 3 angles without yaw and roll, it is written the system of disturbed differential equations compared with the mass center, corresponding to ψ, θ, α, the coefficients are linearized, depend of the gyroscopic momentums created by the stability gyroscopes and the automatic regulations mechanisms for the pitch(tangage) stability [5, 17]. Eliminating θ, α from the system we'll study the equation for ψ in concordance with the regulator characteristics. The object of automatic regulation is the horizontal course of the plane. The important elements of the measurement, control, sensors and with response with inverse reaction to the perturbations that compose the regulator are considered: a gyroscope that measure the pitch(tangage) speed ψ and a gyrotachometer that measure the angular speed $\dot{\psi}$, [5,17]. With sensors and potentiometers help these values are transmitted on the collector plate and transducers and amplifiers are turned into electrical signals, by summary they are transmitted through the input function φ for the output command function to the server $\sigma = -C_1\psi - C_2\dot{\psi} - r\xi$. By mechanical, electromagnetic, hydroelectric and gyroscopic effects, with the reaction parameter ξ determined, conform with the conditions from §3, it is obtain the stability for the null solution.

The mechanical reactions of replay to the control will be transmitted by the commanded stabilizer to the ailerons, shutters (solid or jet type), horizontal empennage, horizontal rudder, to the pitch(tangage) momentum around the Oy axis to converge to zero, considering that the perturbations moments by rolling or yaw be very small; in this way it is obtained the absolute stability of the horizontal course, (fig.1).

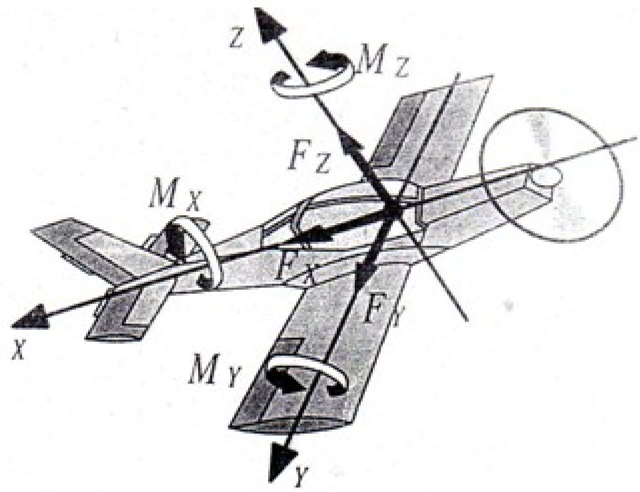

Fig. 1. The momentums in the case of the airplane dynamics.

A. The method of the Liapunov solution for (a.r.a.s.). We'll write the reduce system of equations dimensionless [17], corresponding to the pitch(tangage) perturbation $\psi = x$ in concordance with the functions and characteristics of the regulator connections.

$$\dddot{x} + a_1\ddot{x} + a_2\dot{x} = l\dot{y} + lmy$$

$$\sigma = -c_1x - c_2\dot{x} - r\xi; \psi = x; \dot{\psi} = \frac{dx}{dt} \tag{34}$$

Here, in the constants that appear have been included mass moments, moments of inertia, gyroscopic moments $a_1, a_2, l, m > 0, a_1^2 > 4a_2$ and the characteristic parameters of regulator $c_1, c_2, r > 0, b_2 = l, b_3 = l(m - a_1)$. The right side of the equation is actually the expression of server represented by the nonlinear function $\varphi(\sigma)$. Will write the system (34) with (1)-(4) using the next notations: $x_1 = x = \psi$, $x_2 = \dot{x} = \dot{x}_1 = \dot{\psi}$, $x_3 = \dot{x}_2 - ly$, $y = \xi$, $\dot{y} = \dot{\xi} = \varphi(\sigma)$.

$$\dot{x} = Ax + By, \dot{y} = \dot{\xi} = \varphi(\sigma), \sigma = c'x - r\xi \tag{35}$$

The matrix from (35) is:

$$x=\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix},A=\begin{pmatrix}0&1&0\\0&0&1\\0&-a_2&-a_1\end{pmatrix},$$
$$B=\begin{pmatrix}0\\b_2\\b_3\end{pmatrix},C=\begin{pmatrix}-c_1\\-c_2\\0\end{pmatrix} \tag{35}$$

Using the linear transform:

$$u=AX+B\xi,\dot{\sigma}=C'x-r\xi,u=\begin{pmatrix}u_1\\u_2\\u_3\end{pmatrix} \tag{36}$$

Obtain the simplify system, by derivation:

$$\dot{U}=AU+B\varphi(\sigma),\dot{\sigma}=C'U-r\varphi(\sigma) \tag{37}$$

The system (35) has the unique solution $(x=0,\xi=0)$ and (37) $(U=0,\sigma=0)$. The absolute stability will be realize compare with these null solutions. The characteristic polynomial $P(\lambda)=\det(A-\lambda E)=0$, $\lambda(\lambda^2+a_1\lambda+a_2)=0$ with the notations: $a_1=2p,a_2=q$ has the roots:

$$\lambda_1=-p+\sqrt{p^2-q},\lambda_2=-p-\sqrt{p^2-q}$$
$$\lambda_1<0,\lambda_2<0,\lambda_3=0 \tag{38}$$

After the diagonalization method (9) – (13), will transform the system (37) with $U=Tz$, $T(t_{ij}),i,j=1,2,3$, determining the matrix T with (9) $AT=TJ,J=diagA$, obtaining :

$$T=\begin{pmatrix}\frac{1}{\lambda_1(\lambda_1-\lambda_2)}&-\frac{1}{\lambda_2(\lambda_1-\lambda_2)}&\frac{1}{\lambda_1\lambda_2}\\\frac{1}{\lambda_1-\lambda_2}&-\frac{1}{\lambda_1-\lambda_2}&0\\\frac{\lambda_1}{\lambda_1-\lambda_2}&-\frac{\lambda_2}{\lambda_1-\lambda_2}&0\end{pmatrix}$$
$$T^{-1}=\begin{pmatrix}0&-\lambda_2&1\\0&-\lambda_1&1\\\lambda_1\lambda_2&-(\lambda_1+\lambda_2)&1\end{pmatrix},z=\begin{pmatrix}z_1\\z_2\\z_3\end{pmatrix} \tag{39}$$

$$\dot{z}=Jz+T^{-1}B\varphi(\sigma),\dot{\sigma}=C'Tz-r\varphi(\sigma) \tag{40}$$

The system (40) is equivalent with (35) (36) and has the unique solution $(z=0,\sigma=0)$ and for this solution we study (a.r.a.s), determining the Liapunov function. To build the Liapunov function corresponding to the transformed system (40) $V=V(z,\varphi(\sigma))$, apply the calculus technique presented in (22) – (25) for the special case $\mathrm{Re}(\lambda_{1,2})<0,\lambda_3=0$ at (26), (27). The system (40) became:

$$\dot{z}_1=\lambda_1 z_1+b_1^{'}\varphi(\sigma);\dot{z}_2=\lambda_2 z_2+b_1^{'}\varphi(\sigma),\dot{z}_3=b_3^{'}\varphi(\sigma)$$
$$\dot{\sigma}=f_1z_1+f_2z_2+f_3z_3-r\varphi(\sigma) \tag{41}$$
$$b_1^{'}=b_3-\lambda_2 b_2,b_2^{'}=b_3-\lambda_1 b_2,b_3^{'}=b_3-(\lambda_1+\lambda_2)b_2$$

$$f_1 = -\frac{c_1 + \lambda_1 c_2}{\lambda_1(\lambda_1 - \lambda_2)},\ f_2 = \frac{c_1 + \lambda_2 c_2}{\lambda_2(\lambda_1 - \lambda_2)},\ f_3 = -\frac{c_1}{\lambda_1 \lambda_2}.$$

In this case we choose the Liapunov function conforms to (22), (27)

$$V(z,\sigma) = \frac{1}{2}A_1 z_1^2 + \frac{1}{2}A_2 z_2^2 + \frac{1}{2}A\, z_3^2 + \int_0^\sigma \varphi(\sigma)d\sigma \tag{42}$$

where $A_1, A_2, A > 0$ are fixed, $V(z=0,\sigma=0)=0$ and $V(z,\sigma)$ is positive defined. Compute the derivative $\dot{V}$ associated to the system (41)

$$\dot{V} = \sum_{j=1}^{2} A_j \lambda_j z_j^2 - r\varphi^2 + \sum_{j=1}^{2}(A_j\lambda_j b_j^{'} + f_j) z_j \varphi + (Ab_3^{'} + f_3) z_3 \varphi(\sigma) \tag{43}$$

We observe that taking $A_j = -\frac{1}{\lambda_j} > 0$ the negativity of this form is ensured from the first terms, forcing the cancellation of the last term: $Ab_3' + f_3 = 0$, that means:

$$A = -\frac{f_3}{b_3^{'}} = \frac{c_1}{a_1(b_3 + a_1 b_2)} = \frac{c_1}{a_1 lm} > 0.$$

From

$$\dot{V} = -(z_1^2 + z_2^2) - r\varphi^2 + \sum_{j=1}^{2} \varphi z_j \left(\frac{b_j^{'}}{\lambda_j} - f_j \right) \tag{44}$$

The quadratic form is positive defined for $(-\dot{V})$ in relation with z_1, z_2, φ, with the system (41) or (9). From the Silvester determinant is obtained the necessary and sufficient condition (41) for the rigidity coefficient.

$$r > \left(\frac{b_1^{'}}{\lambda_1} - f_1 \right)^2 + \left(\frac{b_2^{'}}{\lambda_2} - f_2 \right)^2 \tag{45}$$

In this way the characteristic parameters of the regulator r, c_1, c_2 verify the condition (45), ensure the absolute stability of the horizontal fly course of the aircraft. It is observe that in conditions do not appear the function φ, so the nonlinear control function can be choose arbitrary from the admissible class (5), (6).

B. The frequency method for (a.r.a.s.). For this study will applied the frequency method used in §3. because the system (35) is equivalent with (37) and (41), the function $u = -\varphi(\sigma)$ verify the sector condition. By replacing the operator $\frac{d}{dt}$ with the factors is found the transfer function $W(s)$. For simplicity we choose the system (37) with (35), we deduce the transfer function $W(s)$ that is the same for (35) and (41). Applying the Laplace operator in (37) we have:

$$U_1 s = U_2, U_2 s = U_3 + b_2\varphi, U_3 s = -a_2 U_2 - a_1 U_3 + b_3 \varphi$$
$$\sigma s = -c_1 U_1 - c_2 U_2 - r\varphi \tag{46}$$

Eliminating from these relations U_1, U_2, U_3 it is found the connection $\sigma = W(s)(-\varphi)$:

$$W(s) = \frac{1}{s^2}\left(rs + \frac{[b_2(s + a_1) + b_3](c_2 s + c_1)}{s^2 + a_1 s + a_2} \right) \tag{47}$$

We observe that $W(s)$ has a double pole in $s_0 = 0$ and $s_1 = \lambda_1 < 0, s_2 = \lambda_2 < 0$, being in the special case of the frequency method, Criterion3 (a.r.a.s) from §3. next, we verify the conditions from Criterion3.

$$\rho = \lim_{s\to 0} s^2 W(s) = \frac{lmc_1}{a_2} > 0, b_2 = l > 0,$$

$$b_3 = l(m - a_1) > 0, a_1 > 0, a_2 > 0, c_1 > 0 \tag{48}$$

$$\mu = \lim_{s\to 0} \frac{d}{ds}(s^2 W(s)) = r + \frac{l}{a_2^2}\left[c_1(a_1^2 + a_2) - m(a_1c_1 - a_2c_2)\right] > 0 \tag{49}$$

From (49) we obtain conditions for r, m, c_2

$$r > \frac{l}{a_2^2}\left[m(a_1c_1 - a_2c_2) - c_1(a_1^2 + a_2)\right] > 0$$

$$m > \frac{c_1(a_1^2 + a_2)}{a_1c_1 - a_2c_2} > 0, \frac{a_1c_1}{a_2} > c_2 > 0 \tag{50}$$

$$\pi(\omega) = \omega \operatorname{Im} W(j\omega) = -r - l\frac{\omega^2[a_1c_2 - (c_1 + mc_2)] + [a_2(c_1 + mc_2) - a_1c_1(m - a_1)]}{(a_2 - \omega^2)^2 + a_1^2\omega^2} =$$

$$= -r + g(\omega) \tag{51}$$

$$\lim_{\omega\to\infty} \pi(\omega) = -r < 0, \lim_{\omega\to 0} \pi(\omega) = -r + g(0) < 0 \tag{52}$$

From (52) we observe that $r > g(0)$ is from (50) condition. For the rigidity coefficient r we obtain the equivalence with (45). It is observe that by this qualitative criterion are necessary and numerical data in the space of parameters for regulator.

The condition $\pi(\omega) = -r + g(\omega) < 0, \forall \omega \geq 0$ because $g(0) > 0$ is the right member from (50), $g(\omega)$ is derivable, $g'(\omega) < 0$, $\lim_{\omega\to\infty} g(\omega) = 0$ ($g = g(\omega)$ is an even function on $(-\infty, \infty)$ with $g(0)$ maximal.

The automatic regulation for the absolute stability of the horizontal course of flying is presented in the Figs. 2 and 3.

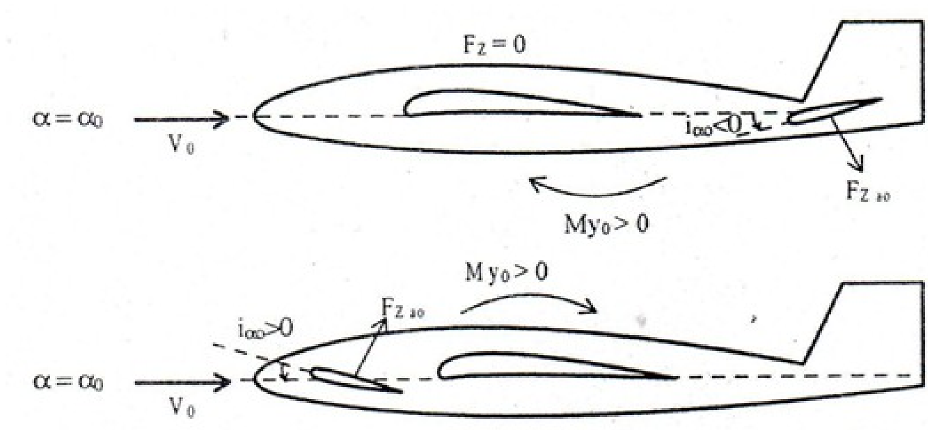

Fig. 2. The scheme of the arrangement for the horizontal empennage compare with the wing.

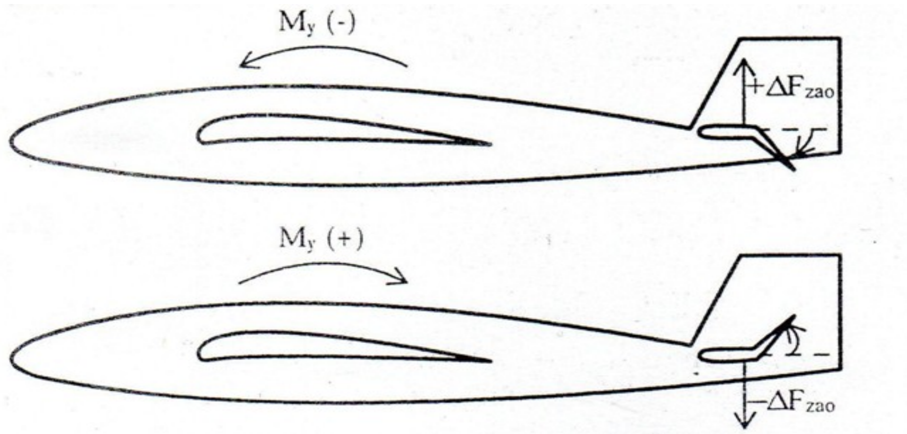

Fig. 3. The command of the pitch momentum.

5. THE ABSOLUTE STABILITY IN THE AUTOMATIC REGULATION OF THE WOOD CUTTING

The high precision of the tools wood cutting with tools machine, implies an automatic regulation of the processes. Here, are modeling and are studying the nonlinear dynamics of the cutting processes (CP) witch tools inside of the wood blocks, the composite materials blocks or hardwood [3].

These (CP) are: CP of drilling, CP of milling, CP of grinding, screw machine, spindle bearing. Machine tool bar is provided with an inner elastic hard wood cutting, cutting inside to run the required geometric rotation and advancing to step slow. Because of the variation in hardness, density, coefficient of elasticity, material composition manufactured by the process disturbances will occur in work mode: transverse vibration due to shaft rotation or longitudinal vibrations to advance. Automatic controller is equipped with sensors, micrometers, tensiometers, rigid response mechanisms of signals output power amplifiers and accelerators. Their purpose is to adjust the characteristics to obtain asymptotic stability of the system work, resulting in high precision components. We will study the two methods described above in §2, §3.

A. The (a.r.a.s.) method by Liapunov function

Consider the dynamic system modeled mathematically, brought to a canonical form of Cauchy, autonomous, with features automatic adjustment for absolute stability of dynamic cutting machining processes. [3, 14]

$$\begin{cases} \dot{x}_1 = a_{11}x_1 + b_1\xi \\ \dot{x}_2 = a_{23}x_3 \\ \dot{x}_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 \\ \dot{x}_4 = a_{44}x_4 + b_4\xi \end{cases}, \sigma = c_2x_2 + c_4x_4 - r\xi, \dot{\xi} = \varphi(\sigma) \quad (25)$$

where a_{ij}, b_i, r, ξ are constants $i, j = 1,2,3,4$.

$$a_{11} = -m < 0, a_{31} = n > 0, a_{32} = -\varepsilon n < 0, a_{33} = -p < 0, a_{44} = -l < 0$$
$$a_{23} = 1.c_2 = 1, c_4 = c < 0, b_1 = b > 0, b_4 = d - r > 0, r > 0 \quad (26)$$

These, according to Lagrange's equations of the parameters are mass produced; mass inertia, elastic constants, strain or pressure coefficients, and σ, r, ξ are the characteristics of the server. We assume that the input function φ is generally nonlinear and check conditions (5) (6). We observe that the linear response function σ of the server control the elements x_2 - the speed of rotation of the cutting bar and x_4 - the speed of advancing its material.

We check the absolute stability of the system solution from zero $(x = 0, \xi = 0)$. Suppose that the block linear system (XA) is asymptotically stable as follows from relations: $\det A \neq 0, \mathrm{Re}(\lambda_i) < 0, i = 1,2,3,4$.

$$P(\lambda) = D(\lambda) = \begin{vmatrix} a_{11}-\lambda & 0 & 0 & 0 \\ 0 & -\lambda & a_{23} & 0 \\ a_{31} & a_{32} & a_{33}-\lambda & 0 \\ 0 & 0 & 0 & a_{44}-\lambda \end{vmatrix} =$$

$$= (a_{11}-\lambda)(a_{44}-\lambda)(\lambda^2 - \lambda a_{33} - a_{23}a_{32}) = 0 \tag{27}$$

$$\lambda_1 = a_{11} = -m < 0, \lambda_4 = a_{44} = -l < 0, \lambda_{2,3} = \frac{1}{2}\left(-p \pm \sqrt{p^2 - 4\varepsilon n}\right) < 0, \lambda_i \in \mathbf{R} \tag{28}$$

In this case, following the diagonalization method §2 with the formulas (9) - (13) or directly choose the option remark (R_2) we get the diagonal system in z_i and $\dot{\sigma}$ (12’), (13’):

$$\dot{z}_i = \lambda_i z_i + \varphi(\sigma); \dot{\sigma} = \sum_{i=1}^{4} f_i z_i - r\varphi(\sigma), i = 1,...,4 \tag{29}$$

$$f_1 = \frac{b_1 a_{31}}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, f_2 = \frac{-b_1 a_{31}}{(\lambda_1-\lambda_2)(\lambda_2-\lambda_3)}, f_1 = \frac{b_1 a_{31}}{(\lambda_1-\lambda_3)(\lambda_2-\lambda_3)}, f_4 = b_4 c_4 < 0 \tag{30}$$

We observe that $f_1 + f_2 + f_3 = 0$ and whatever is the choice of order quantities $\lambda_1, \lambda_2, \lambda_3$ are strictly negative, always two of the functions $f_i, i = 1,2,3$ have the same sign and the third function takes opposite sign using relation (26). In this case the stability of the following (29) from the null solution ($z_i = 0, \sigma = 0$) and that $f_{4<0}$ we can construct such Lyapunov function:

$$V(z,\sigma) = -\frac{1}{2} f_4 z_4^2 - \frac{1}{2}\sum_{i=1}^{3} \frac{a_i^2 z_i^2}{\lambda_i} - \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{a_i a_j}{\lambda_i + \lambda_j} z_i z_j + \int_0^{\sigma} \varphi(\sigma) d\sigma \tag{31}$$

where the real coefficients a_1, a_2, a_3 will be determined.

From $\lambda_i < 0, \lambda_i + \lambda_j < 0, V(0) = 0$, the summative terms after $i = 1,2,3$ determine a positive quadratic form positive definite and the integral positive term, we have $V(z,\sigma) > 0$ allowed in the vicinity.

We calculate $\dot{V}(z,\sigma)$ attach to the system (29), and obtaining:

$$\dot{V}(z,\sigma) = -f_4 \lambda_4 z_4^2 - (a_1 z_1 + a_2 z_2 + a_3 z_3)^2 - \varphi \sum_{i=1}^{3} z_i \left(\frac{a_i^2}{\lambda_i} + \sum_{j \neq i} \frac{2 a_i a_j}{\lambda_i + \lambda_j} - f_i \right) \tag{32}$$

Observe that $\dot{V}(z = 0, \sigma = 0) = 0$ and to have the strict negativity it must that the parenthesis from the term φ to be null

$$\frac{a_i^2}{\lambda_i}+\sum_{j\neq i}\frac{2a_ia_j}{\lambda_i+\lambda_j}-f_i=0,i,i=1,2,3 \qquad \text{(S*)} \qquad (33)$$

The system (S*) $F_i(a_1,a_2,a_3)=0, i=1,2,3$ is implicit with three equations with three unknowns and the existence of solutions is provided by the system Jacobian $J=\frac{D(F_1,F_2,F_3)}{D(a_1,a_2,a_3)}\neq 0$. A helping calculation prove that if each equation from (33) is multiplying respectively by $\frac{1}{\lambda_i}$ and summing, it is obtain:

$$\Gamma^2=\left(\sum_{i=1}^{3}\frac{a_i}{\lambda_i}\right)^2=\sum_{i=1}^{3}\frac{f_i}{\lambda_i}=S>0,\Gamma=\pm\sqrt{S} \qquad (34)$$

A conditions that indicates that the knowing sum (S) is strictly positive and in the parametric space (a_1,a_2,a_3) the symmetrical plane (π_{12}) , $\sum_{i=1}^{3}\frac{a_i}{\lambda_i}=\pm\sqrt{S}$ where exists a solution, don't admit the null solution because $f_i\neq 0$. Is obtained as:

$$J=\pm 8\frac{\sqrt{S}(a_1+a_2+a_3)[a_1(\lambda_2+\lambda_3)+a_2(\lambda_1+\lambda_3)+a_3(\lambda_1+\lambda_2)]}{\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2)(\lambda_1+\lambda_3)(\lambda_2+\lambda_3)}\neq 0 \qquad (35)$$

Because the two factors from the numerator parenthesis are planes passing through the origin, the solution is contained in the planes (π_{12}). Analyzing the system (S*) after the sign of $f_i,(a_1,a_2,a_3)$ these solutions from (π_{12}) are not in the I,V octant. If all would have the same sign than $f_i>0$. We proved that exists solutions of the system (33) and $\dot{V}<0$; results that the Liapunov function provide the automatic regulation of the absolute stability. For this application and sufficient conditions of type (15), (16) with numerical data, are obtain.

B. The frequency method for (a.r.a.s.)

For this study will applied the frequency method used in §3. Because the system (25) is equivalent with (39) the function $u=-\varphi(s)$ verify the sector conditions. By replacing the operator $\frac{d}{dt}$ with the factor s is found the transfer function $W(s)$. So, from (29) is obtain for $\sigma=W(s)(-\varphi)$

$$sz_i=\lambda_iz_i+\varphi; s\sigma=\sum_{i=1}^{4}f_iz_i-r\varphi \qquad (35)$$

Eliminating from (35) z_i we obtain the transfer function from $\sigma=W(s)(-\varphi)$

$$W(s)=\frac{1}{s}\left(r-\sum_{i=1}^{4}\frac{\lambda_if_i}{s-\lambda_i}\right) \qquad (36)$$

Because the real roots $s_i=\lambda_i$ verify $\mathrm{Re}(\lambda_i)<0$ the transfer function has a simple pole in $s=0$ and the rest of real roots with $\mathrm{Re}(s_i)<0$. In this case we have the Criterion II of critical singularity from §3 for (a.r.a.s.). here, the conditions (15), (19) and II a), b) are verified fro the method A and must verified the condition c). So, $\rho = \lim_{s\to 0} sW(s) = r + \sum_{i=1}^{4} f_i = r + f_4 = r + b_4 c_4 > 0$ implies $r + c(d-r) > 0$ that mean $r > \frac{cd}{1-c} > 0, d < 0$. From $W(s = j\omega) = U(\omega) + jV(\omega)$, we have:

$$U(\omega) = \sum_{i=1}^{4} \frac{f_i \lambda_i}{\lambda_i^2 + \omega^2}, \quad V(\omega) = -\frac{r}{\omega} + \frac{1}{\omega} \sum_{i=1}^{4} \frac{\lambda_i^2 f_i}{\lambda_i^2 + \omega^2}$$

For given $k > 0$, from the condition $0 < \varphi(s) < k\sigma$ with $\varphi(\sigma)$ specified, it can be determine $q \in \mathbf{R}$ verifying the condition (24'). The parameters λ_i, f_i are knowing from (28), (30), the nonlinear function φ is chosen with σ from (25) and for specified numerical data determine k and the delimitation of q. The existence of these conditions can be perform hodographically for (a.r.a.s.) at this application.

4. CONCLUSIONS

The importance of this paper is evident in the fact that the problem of absolute stability is systematized by the two methods. It is remark that fact that the application regarding (a.r.a.s.) for the horizontal fly course with automatic pilot is studied for the critical difficult cases, when the roots of characteristic polynomial or the pole of transfer function is in origin (on the imaginary axis).

For the Liapunov function building we applied an original method. For another studies are recommend the published results of the researchers [1, 15, 19, 20, 11].

REFERENCES

[1] Barbasin, E., *Liapunov's function*, Ed. Nauka, Moscov, 1970.

[2] Belea, C., *The system theory - nonlinear system*, Ed. Did. Ped., Bucuresti, 1985.

[3] Chiriacescu, S., *Stability in the Dynamics of Metal Cutting*, Ed. Elsevier, 1990; *Dynamics of cutting machines*, Ed. Tehnica, Bucuresti, 2004 (in Romanian).

[4] Dumitrache, I., *The engineering of automatic regulation*, Ed. Politehnica Press, Bucuresti, 2005.

[5] Lungu, R., *Gyroscopic equipment and systems*, Ed. Press Universitaria, Craiova, 1997.

[6] Lupu, C., et.al., *Industrial process control systems*, Ed. Printech, Bucuresti, 2004.

[7] Lupu, C., et.al., P*ractical solution for nonlinear process control*, Ed. Politehnica Press, Bucuresti, 2010.

[8] Lupu, M., Florea, O., Lupu, C., *Ovidius University Annals Series: Civil Engineering*, **1**(11), 63, 2009.

[9] Lupu, M., Isaia, F., *J. Creative Math*, **16**, 81, 2007.

[10] Lupu, M., Isaia, F., *Analele Univ. Bucuresti*, **LV**, 203, 2006.

[11] Lupu, M., Florea, O., Lupu, C., *Romai Journal*, **6**(2), 185, 2010.

[12] Lupu, M., Scheiber, E., Florea, O., Lupu, C., *Journal of Computational and Applied Mechanics*, **12**(1), 2011.

[13] Lurie, A. Y., *Nonlinear problems from the automatic control*, Ed. Gostehizdat Moskow, 1951.

[14] Merkin, D. R., *Introduction in the movement stability theory*, Ed. Nauka, Moskow, 1987.

[15] Nalepin, R. A. et.al., *Exact methods for the nonlinear system control in case of automatic regulation*, Ed. M.S. Moskva, 1971.

[16] Popescu, D. et.al., *Modéllisation, Identication et Commande des Systemes*, Ed.Acad. Romaine, Bucuresti, 2004.

[17] Popov, E. P., *Applied theory of control of nonlinear systems*, Ed. Nauka, 1973; Verlag Technik, Berlin 1964.

[18] Popov, V. M., *The hypersensitivity of automatic systems*, Ed. Acad Romane, 1966; Ed. Springer -Verlag, 1973; Ed. Nauka, 1970.

[19] Rasvan, V., *Theory of Stability*, Ed. St. Enciclopedica, Bucuresti, 1987.

[20] Rouche, N., Habets, P., Leloy, M., *Stability theory by Liapunov's Direct Method*, Springer Verlag, 1977.