ORIGINAL PAPER

A REFINEMENT OF HOLDER'S AND ITS REVERSE INEQUALITY USING AN INEQUALITY OF N. MINCULETE

LOREDANA CIURDARIU[1]

Manuscript received: 12.07.2012; Accepted paper: 16.08.2012;

Published online: 15.09.2012.

Abstract: *In this paper we will give two refinements for the Holder's inequality and its reverse using two refinements of the Kittaneh-Manasrah inequality presented by N. Minculete.*

Keywords: *Young's inequality, Kittaneh-Manasrah's inequality, Holder's inequalit.*

2010 Mathematics subject classification: *26D15.*

1. INTRODUCTION

In order to prove the results of the following sections, we need to recall the next properties.

Theorem 2.1. ([4]) For $a,b \geq 1$ and $\lambda \in (0,1)$, we have

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)\log^2\left(\frac{a}{b}\right),$$

where $r = \min\{\lambda, 1-\lambda\}$, $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

Application 3.2. ([4]) For $0 < a,b \leq 1$ and $\lambda \in (0,1)$, we have

$$r(\sqrt{a}-\sqrt{b})^2 + A(\lambda)ab\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^{\lambda}b^{1-\lambda} \leq$$

$$\leq (1-r)(\sqrt{a}-\sqrt{b})^2 + B(\lambda)ab\log^2\left(\frac{a}{b}\right),$$

where $r, A(\lambda), B(\lambda)$ are given in Theorem 2.1.

2. THE RESULTS

Next result will present a variant of reverse of the Holder inequality in some particular cases, using as a starting point Theorem 2.1.

[1] Politehnica University of Timisoara, 300006 Timisoara, Romania. E-mail: clorsim61@gmail.com.

Theorem 1. Let $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$. If f and g are two positive functions which admit integral on [a,b] for which there exist $\int_a^b f^p(x)dx$ and $\int_a^b g^q(x)dx$ finite with $\int_a^b f^p(x)dx > 0, \int_a^b g^q(x)dx > 0$ and $\frac{g^q(x)}{\int_a^b g^q(x)dx} \le \frac{f^p(x)}{\int_a^b f^p(x)dx} \le M \frac{g^q(x)}{\int_a^b g^q(x)dx}, (\forall)x \in [a,b], M > 1$

then we have:

$$1 - \frac{\int_a^b f(x)\cdot g(x)dx}{(\int_a^b f^p(x)dx)^{1/p}(\int_a^b g^q(x)dx)^{1/q}} \le \frac{2}{\min\{p,q\}}\left(1 - \frac{\int_a^b f^{p/2}(x)g^{q/2}(x)dx}{(\int_a^b f^p(x)dx)^{1/2}(\int_a^b g^q(x)dx)^{1/2}}\right) + \\ + \left(\frac{1}{2pq} - \frac{1}{4\min\{p,q\}}\right)\log^2 M$$

Proof: Taking in Theorem 2.1, b=1, $\lambda = \frac{1}{p}, 1 - \lambda = \frac{1}{q}$ we will obtain:

$$\frac{1}{p}a_1 + \frac{1}{q} - a_1^{1/p} \le (1 - \frac{1}{\max\{p,q\}})(\sqrt{a_1} - 1)^2 + \left(\frac{1}{2pq} - \frac{1}{4}(1 - \frac{1}{\max\{p,q\}})\right)\log^2(a_1) \le \\ \le (1 - \frac{1}{\max\{p,q\}})(\sqrt{a}_1 - 1)^2 + \left(\frac{1}{2pq} - \frac{1}{4}(1 - \frac{1}{\max\{p,q\}})\right)\log^2(M)$$

and then putting $a_1 = \frac{f^p}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q} \ge 1$ we will have,

$$\frac{1}{p}\frac{f^p}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q} + \frac{1}{q} - \frac{f}{(\int_a^b f^p(x)dx)^{1/p}}\frac{(\int_a^b g^q(x)dx)^{1/p}}{g^{q/p}} \le \\ \le (1 - \frac{1}{\max\{p,q\}})\left(\frac{f^p}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q} - 2\frac{f^{p/2}}{(\int_a^b f^p(x)dx)^{1/2}}\frac{(\int_a^b g^q(x)dx)^{1/2}}{g^{q/2}} + 1\right) + \\ + \left(\frac{1}{2pq} - \frac{1}{4}(1 - \frac{1}{\max\{p,q\}})\right)\log^2(M).$$

By calculus we obtain,

$$\frac{1}{p}\frac{f^p}{\int_a^b f^p(x)dx} + \frac{1}{q}\frac{g^q}{\int_a^b g^q(x)dx} - \frac{f}{(\int_a^b f^p(x)dx)^{1/p}}\frac{g}{(\int_a^b g^q(x)dx)^{1/q}} \le$$

$$\leq (1-\frac{1}{\max\{p,q\}})\left(\frac{f^{p}}{\int_a^b f^{p}(x)dx}-2\frac{f^{p/2}g^{q/2}}{(\int_a^b f^{p}(x)dx)^{1/2}(\int_a^b g^{q}(x)dx)^{1/2}}+\frac{g^{q}}{\int_a^b g^{q}(x)dx}\right)+$$

$$+\left(\frac{1}{2pq}-\frac{1}{4}(1-\frac{1}{\max\{p,q\}})\right)\log^{2}(M)\frac{g^{q}}{\int_a^b g^{q}(x)dx}.$$

and then by integration the last inequality becomes:

$$\frac{1}{p}+\frac{1}{q}-\frac{\int_a^b f(x)g(x)dx}{(\int_a^b f^{p}(x)dx)^{1/p}(\int_a^b g^{q}(x)dx)^{1/q}}\leq (1-\frac{1}{\max\{p,q\}})\left(2-2\frac{\int_a^b f^{p/2}(x)g^{q/2}(x)dx}{(\int_a^b f^{p}(x)dx)^{1/2}(\int_a^b g^{q}(x)dx)^{1/2}}\right)+$$

$$+\left(\frac{1}{2pq}-\frac{1}{4}(1-\frac{1}{\max\{p,q\}})\right)\log^{2}(M).$$

Using that $1-\frac{1}{\max\{p,q\}}=\frac{1}{\min\{p,q\}}$ and that $\frac{1}{p}+\frac{1}{q}=1$ previous inequality will be:

$$1-\frac{\int_a^b f(x)g(x)dx}{(\int_a^b f^{p}(x)dx)^{1/p}(\int_a^b g^{q}(x)dx)^{1/q}}\leq \frac{2}{\min\{p,q\}}\left(1-\frac{\int_a^b f^{p/2}(x)g^{q/2}(x)dx}{(\int_a^b f^{p}(x)dx)^{1/2}(\int_a^b g^{q}(x)dx)^{1/2}}\right)+$$

$$+\left(\frac{1}{2pq}-\frac{1}{4\min\{p,q\}}\right)\log^{2}(M).$$

Now we consider (Ω, F, μ) a measure space and p a real number with $p \geq 1$ then the space $L^{p}=L^{p}(\Omega,F,\mu)$ is the collection of all complex-valued Borel measurable functions f such that

$$\int_{\Omega}|f|^{p}\,d\mu<\infty \text{ and } \|f\|_{p}=(\int_{\Omega}|f|^{p}\,d\mu)^{1/p},\, f\in L^{p}.$$

Using the same method we can prove the following result:

Theorem 2. Let $1<p<\infty$, $1<q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. If f and g are two positive functions $f\in L^{p}, g\in L^{q}$ with $\|f\|_{p}>0, \|g\|_{q}>0$ for which there exist

$$\frac{g^{q}}{\|g\|_{q}^{q}}\leq\frac{f^{p}}{\|f\|_{p}^{p}}\leq M\,\frac{g^{q}}{\|g\|_{q}^{q}},M>1$$

then

$$1-\frac{\|f\cdot g\|_{1}}{\|f\|_{p}\|g\|_{q}}\leq\frac{2}{\min\{p,q\}}\left(1-\frac{\int_{\Omega}f^{p/2}g^{q/2}d\mu}{\|f\|_{p}^{p/2}\|g\|_{q}^{q/2}}\right)+\left(\frac{1}{2pq}-\frac{1}{4\min\{p,q\}}\right)\log^{2}M.$$

Consequence 3. Let $1 < p < \infty, 1 < q < \infty, \frac{1}{p} + \frac{1}{q} = 1$. If *f* and *g* are two strict positive and continuous functions for which $\frac{g^q(x)}{\int_a^b g^q(x)dx} \le \frac{f^p(x)}{\int_a^b f^p(x)dx}, (\forall) x \in [a,b]$ then,

$$1 - \frac{\int_a^b f(x) \cdot g(x)dx}{(\int_a^b f^p(x)dx)^{1/p} (\int_a^b g^q(x)dx)^{1/q}} \le \frac{2}{\min\{p,q\}} \left(1 - \frac{\int_a^b f^{p/2}(x) g^{q/2}(x)dx}{(\int_a^b f^p(x)dx)^{1/2} (\int_a^b g^q(x)dx)^{1/2}} \right) +$$

$$+ \left(\frac{1}{2pq} - \frac{1}{4\min\{p,q\}} \right) \log^2 M,$$

where $M = \frac{\int_a^b g^q(x)dx}{\int_a^b f^p(x)dx} \sup_{x \in [a,b]} \frac{f^p(x)}{g^q(x)}$.

Proof: The functions f and g being strict positive and continuous so will be f^p, g^q and $\frac{f^p}{g^q}$ and then $\sup_{x \in [a,b]} \frac{f^p(x)}{g^q(x)}$ exists and is finite.

Theorem 4. Let $1 < p < \infty,\ 1 < q < \infty,\ \frac{1}{p} + \frac{1}{q} = 1$. If f and g are two positive functions which admit integral on [a,b] for which there exist $\int_a^b f^p(x)dx$ and $\int_a^b g^q(x)dx$ and are finite with $\int_a^b f^p(x)dx > 0, \int_a^b g^q(x)dx > 0$, and

$$m \frac{g^q(x)}{\int_a^b g^q(x)dx} < \frac{f^p(x)}{\int_a^b f^p(x)dx} \le \frac{g^q(x)}{\int_a^b g^q(x)dx}, (\forall) x \in [a,b], 0 < m < 1$$

then we have:

$$1 - \frac{\int_a^b f(x) \cdot g(x)dx}{(\int_a^b f^p(x)dx)^{1/p} (\int_a^b g^q(x)dx)^{1/q}} \le \frac{2}{\min\{p,q\}} \left(1 - \frac{\int_a^b f^{p/2}(x) g^{q/2}(x)dx}{(\int_a^b f^p(x)dx)^{1/2} (\int_a^b g^q(x)dx)^{1/2}} \right) +$$

$$+ \left(\frac{1}{2pq} - \frac{1}{4\min\{p,q\}} \right) \log^2 (\frac{1}{m})$$

Proof: We put $\lambda = \frac{1}{q}, 1 - \lambda = \frac{1}{p}, a_1 = 1$, and then we obtain for $0 < b_1 \le 1$ in Application 3.2 the following inequality:

$$\frac{1}{q}+\frac{1}{p}b_1-b_1^{1/p}\le\frac{1}{\min\{p,q\}}(1-\sqrt{b_1})^2+(\frac{1}{2pq}-\frac{1}{4\min\{p,q\}})b_1\log^2(\frac{1}{b_1})\le$$
$$\le\frac{1}{\min\{p,q\}}(1+b-2\sqrt{b})+(\frac{1}{2pq}-\frac{1}{4\min\{p,q\}})b_1\log^2(\frac{1}{m}),$$

taking into account that if

$$b_1=\frac{f^p}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q} \text{ then } 0<m<b_1\le 1 \text{ that is } 1\le\frac{1}{b_1}<\frac{1}{m}$$

and then $\log^2(\frac{1}{b})<\log^2(\frac{1}{m})$.

Previous inequality becomes:

$$\frac{1}{q}\frac{g^q}{\int_a^b g^q(x)dx}+\frac{1}{p}\frac{f^p}{\int_a^b f^p(x)dx}-\frac{f}{(\int_a^b f^p(x)dx)^{1/p}}\frac{g}{(\int_a^b g^q(x)dx)^{1/q}}\le$$
$$\le\frac{1}{\min\{p,q\}}(\frac{g^q}{\int_a^b g^q(x)dx}+\frac{f^p}{\int_a^b f^p(x)dx}-2\frac{f^{p/2}g^{q/2}}{(\int_a^b f^p(x)dx)^{1/2}(\int_a^b g^q(x)dx)^{1/2}})+$$
$$+(\frac{1}{2pq}-\frac{1}{4\min\{p,q\}})\frac{f^p}{\int_a^b f^p(x)dx}\log^2(\frac{1}{m}).$$

Then by integration from a to b we obtain the desired inequality.

Consequence 5. Let $1<p<\infty,\ 1<q<\infty,\ \frac{1}{p}+\frac{1}{q}=1$. If f and g are two positive functions which admit integral on [a,b] for which there exist $\int_a^b f^p(x)dx$ and $\int_a^b g^q(x)dx$ and are finite with $\int_a^b f^p(x)dx>0,\int_a^b g^q(x)dx>0$, and

$$m\frac{g^q(x)}{\int_a^b g^q(x)dx}<\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{q}{p}\frac{s}{t}\le\frac{g^q(x)}{\int_a^b g^q(x)dx},(\forall)x\in[a,b],0<m<\frac{q}{p}\frac{s}{t}\le 1,t,s>0$$

then we have:

$$\frac{t}{q^2}+\frac{s}{p^2}-\frac{\int_a^b f(x)\cdot g(x)dx}{(\int_a^b f^p(x)dx)^{1/p}(\int_a^b g^q(x)dx)^{1/q}}\left(\frac{t}{q}\right)^{1/q}\left(\frac{s}{p}\right)^{1/p}\le$$
$$\le\frac{1}{\min\{p,q\}}\left(\frac{t}{q}+\frac{s}{p}-2\frac{\int_a^b f^{p/2}(x)g^{q/2}(x)dx}{(\int_a^b f^p(x)dx)^{1/2}(\int_a^b g^q(x)dx)^{1/2}}\left(\frac{s}{p}\frac{t}{q}\right)^{1/2}\right)+$$

$$+\left(\frac{1}{2pq}-\frac{1}{4\min\{p,q\}}\right)\frac{s}{p}\log^2(\frac{1}{m}).$$

Proof: As in the proof of previous theorem we take $\lambda=\frac{1}{q}, 1-\lambda=\frac{1}{p}, a_1=1$, and then we obtain for

$$0<b_1=\frac{f^p}{\int_a^b f^p(x)dx}\frac{\int_a^b g^q(x)dx}{g^q}\frac{q}{p}\frac{s}{t}\leq 1$$

the inequality.

Consequence 6. Let $1<p<\infty$, $1<q<\infty$, $\frac{1}{p}+\frac{1}{q}=1$. If f and g are two positive functions which admit integral on [a,b] for which there exist $\int_a^b f^p(x)dx$ and $\int_a^b g^q(x)dx$ and are finite with $\int_a^b f^p(x)dx>0, \int_a^b g^q(x)dx>0$, and

$$\frac{g^q(x)}{\int_a^b g^q(x)dx}<\frac{f^p(x)}{\int_a^b f^p(x)dx}\frac{q}{p}\frac{s}{t}\leq\frac{g^q(x)}{\int_a^b g^q(x)dx}M, (\forall)x\in[a,b], 1<\frac{q}{p}\frac{s}{t}\leq M, t,s>0$$

then we have:

$$\frac{t}{q^2}+\frac{s}{p^2}-\frac{\int_a^b f(x)\cdot g(x)dx}{(\int_a^b f^p(x)dx)^{1/p}(\int_a^b g^q(x)dx)^{1/q}}\left(\frac{t}{q}\right)^{1/q}\left(\frac{s}{p}\right)^{1/p}\leq$$

$$\leq\frac{1}{\min\{p,q\}}\left(\frac{t}{q}+\frac{s}{p}-2\frac{\int_a^b f^{p/2}(x)g^{q/2}(x)dx}{(\int_a^b f^p(x)dx)^{1/2}(\int_a^b g^q(x)dx)^{1/2}}\left(\frac{s}{p}\frac{t}{q}\right)^{1/2}\right)+$$

$$+\left(\frac{1}{2pq}-\frac{1}{4\min\{p,q\}}\right)\frac{s}{p}\log^2(M).$$

REFERENCES

[1] Ash, B., Measure, R., *Integration and Functional Analysis*, Academic Press, New York and London, 1970.
[2] Kittaneh, F., Manasrah, Y., *J. Math. Anal. Appl.*, **361,** 262, 2010.
[3] Kittaneh, F., Manasrah Y., *Linear and Multilinear Algebra*, **59**(9), 1031, 2011.
[4] Minculete, N., *Creative Math. & Inf.*, **20**(2), 157, 2011.