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SOME GEOMETRIC INEQUALITIES OF RADON-ERDÖS-MORDELL TYPE

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Abstract. *Some Erdös-Mordell type inequalities for general convex poligons are presented.The main tool in the proofs is the Radon inequality.*

Keywords: *Erdös-Mordell type inequality, Radon inequality, convex polygon.*

1. INTRODUCTION

The purpose of this article is to establish some geometric inequalities (other than [2]) on *Erdös-Mordell* - type , in convex polygons, used *J. Radon*'s inequality (see for example[1]).

Let $a,b,c,d,x_k,y_k \in R_+^*, \forall k=\overline{1,n}$ and $X_n=\sum_{k=1}^{n} x_k, Y_n=\sum_{k=1}^{n} y_k$.

Theorem 1. (A generalization of J. *Radon*'s inequality).

If $m,p,q,s \in R_+, r\in[1,\infty)$ *such that* $cY_n^s > d\max_{1\le k\le n} y_k^s, \forall k=\overline{1,n}$ *, then*:

$$\sum_{k=1}^{n}\frac{\left(aX_n^p+bx_k^q\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\ge\frac{\left(an^qX_n^{p+r}+bX_n^{q+r}\right)^{m+1}}{\left(cn^s-d\right)^m Y_n^{m(s+1)}}\cdot\frac{1}{n^{(m+1)(q+r-1)-ms}} \tag{1}$$

Proof: We denoted

$$u_k=\left(aX_n^p+bx_k^q\right)x_k^r, v_k=\left(cY_n^s-dy_k^s\right)y_k, \forall k=\overline{1,n}\ , V_n=\sum_{k=1}^{n}v_k$$

and the LHS of (1) becomes:

$$\sum_{k=1}^{n}\frac{u_k^{m+1}}{v_k^m}=\sum_{k=1}^{n}v_k\left(\frac{u_k}{v_k}\right)^{m+1}=V_n\sum_{k=1}^{n}\frac{v_k}{V_n}\left(\frac{u_k}{v_k}\right)^{m+1}$$

Since the function $f:R_+^*\to R_+^*, f(x)=x^{m+1}$ is convex on R_+^* , we use *Jensen*'s inequality and we obtain that:

$$\sum_{k=1}^{n}\frac{v_k}{V_n}f\left(\frac{u_k}{v_k}\right)\ge\sum_{k=1}^{n}f\left(\frac{v_k}{V_n}\cdot\frac{u_k}{v_k}\right)=\sum_{k=1}^{n}f\left(\frac{u_k}{V_n}\right)=\frac{\left(\sum_{k=1}^{n}u_k\right)^{m+1}}{V_n^{m+1}}\Leftrightarrow\sum_{k=1}^{n}\frac{v_k}{V_n}\left(\frac{u_k}{v_k}\right)^{m+1}\ge\frac{\left(\sum_{k=1}^{n}u_k\right)^{m+1}}{V_n^{m+1}}$$

Therefore,

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$$\sum_{k=1}^{n}\frac{u_k^{m+1}}{v_k^m}\ge V_n\cdot\frac{\left(\sum_{k=1}^{n}u_k\right)^{m+1}}{V_n^{m+1}}=\frac{\left(\sum_{k=1}^{n}u_k\right)^{m+1}}{V_n^m}\Leftrightarrow$$

$$\Leftrightarrow\sum_{k=1}^{n}\frac{\left(aX_n^p+bx_k^q\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\ge\frac{\left(\sum_{k=1}^{n}\left(aX_n^p+bx_k^q\right)x_k^r\right)^{m+1}}{\left(\sum_{k=1}^{n}\left(cY_n^s-dy_k^s\right)y_k\right)^m}=$$

$$=\frac{\left(\sum_{k=1}^{n}\left(aX_n^px_k^r+bx_k^{q+r}\right)\right)^{m+1}}{\left(\sum_{k=1}^{n}\left(cY_n^sy_k-dy_k^{s+1}\right)\right)^m}=\frac{\left(aX_n^p\sum_{k=1}^{n}x_k^r+b\sum_{k=1}^{n}x_k^{q+r}\right)^{m+1}}{\left(cY_n^s\sum_{k=1}^{n}y_k-d\sum_{k=1}^{n}y_k^{s+1}\right)^m}=\frac{\left(aX_n^p\sum_{k=1}^{n}x_k^r+b\sum_{k=1}^{n}x_k^{q+r}\right)^{m+1}}{\left(cY_n^{s+1}-d\sum_{k=1}^{n}y_k^{s+1}\right)^m}$$

Because the functions $g,h,k:R_+^*\to R_+^*, g(x)=x^r, h(x)=x^{q+r}, k(y)=y^{s+1}$ are convex on R_+^* , also by *Jensen's* inequality we have:

$$\sum_{k=1}^{n}x_k^r=\sum_{k=1}^{n}g(x_k)\ge ng\left(\frac{1}{n}\cdot\sum_{k=1}^{n}x_k\right)=n\cdot\frac{X_n^r}{n^r}=\frac{X_n^r}{n^{r-1}},$$

$$\sum_{k=1}^{n}x_k^{q+r}=\sum_{k=1}^{n}h(x_k)\ge nh\left(\frac{1}{n}\cdot\sum_{k=1}^{n}x_k\right)=n\cdot\frac{X_n^{q+r}}{n^{q+r}}=\frac{X_n^{q+r}}{n^{q+r-1}},$$

$$\sum_{i=1}^{n}y_i^{s+1}=\sum_{i=1}^{n}k(y_i)\ge nk\left(\frac{1}{n}\cdot\sum_{i=1}^{n}y_i\right)=n\cdot\frac{Y_n^{s+1}}{n^{s+1}}=\frac{Y_n^{s+1}}{n^s}.$$

Then, we deduce that:

$$\sum_{k=1}^{n}\frac{\left(aX_n^p+bx_k^q\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\ge\frac{\left(a\cdot\frac{X_n^{p+r}}{n^{r-1}}+b\cdot\frac{X_n^{q+r}}{n^{q+r-1}}\right)^{m+1}}{\left(cY_n^{s+1}-\frac{dY_n^{s+1}}{n^s}\right)^m}=$$

$$=\frac{\left(an^qX_n^{p+r}+bX_n^{q+r}\right)^{m+1}}{\left(cn^s-d\right)^mY_n^{m(s+1)}}\cdot\frac{n^{ms}}{n^{(m+1)(q+r-1)}}=\frac{\left(an^qX_n^{p+r}+bX_n^{q+r}\right)^{m+1}}{\left(cn^s-d\right)^mY_n^{m(s+1)}}\cdot\frac{1}{n^{(m+1)(q+r-1)-ms}},$$

and we are done. ■

Observation 1.1. *If* $p=q=s=0$ *, then (1) becomes:*

$$\sum_{k=1}^{n}\frac{(a+b)^{m+1}x_k^{r(m+1)}}{(c-d)^my_k^m}\ge\frac{(a+b)^{m+1}X_n^{r(m+1)}}{(c-d)^mY_n^m}\cdot\frac{1}{n^{(m+1)(r-1)}}\Leftrightarrow\sum_{k=1}^{n}\frac{x_k^{r(m+1)}}{y_k^m}\ge\frac{X_n^{r(m+1)}}{Y_n^mn^{(m+1)(r-1)}}\qquad(1')$$

If we consider r = 1, then by (1') we obtain:

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{y_k^m}\ge\frac{X_n^{m+1}}{Y_n^m}\qquad\textbf{(R)}$$

i.e that is just the inequality of J. Radon, with equality if and only if there exists $t\in R_+^*$ *such that* $x_k=ty_k,\forall k=\overline{1,n}$.

Observation 1.2. *If m = 1, then* (1) *becomes*:

$$\sum_{k=1}^{n}\frac{\left(aX_n^p+bx_k^q\right)^2 x_k^{2r}}{\left(cY_n^s-dy_k^s\right)y_k}\geq\frac{\left(an^qX_n^{p+r}+bX_n^{q+r}\right)^2}{\left(cn^s-d\right)Y_n^{s+1}}\cdot\frac{1}{n^{2(q+r-1)-s}}\tag{1''}$$

If we take $p=q=s=0, r=1$ *, then by (1") we obtain*:

$$\sum_{k=1}^{n}\frac{x_k^2}{y_k}\geq\frac{X_n^2}{Y_n}\tag{B}$$

but that is just the inequality of H. Bergström.

Next, we consider $A_1A_2...A_n$ $(n\geq 3)$, a convex polygon and for any point M from inside the polygon we use the notations: $x_k = MA_k$, y_k the distance from M to the line A_kA_{k+1}, a_k the length of the side $[A_kA_{k+1}]$ of the polygon $\left(\forall k=\overline{1,n}\right)$, $2p$ is the perimeter of the polygon, S is the area of the given polygon and $A_{n+1}\equiv A_1$.

Theorem 2. *If* $A_1A_2...A_n$ $(n\geq 3)$ *is a convex polygon where we use the above notations and*

$$a,b,c,d\in R_+^*\ m,p,q,s\in R_+, r\in[1,\infty),$$

such that

$$cY_n^s>d\max_{1\leq k\leq n}y_k^s, \forall k=\overline{1,n},$$

then:

$$\sum_{k=1}^{n}\frac{\left(aX_n^p+bx_k^q\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\geq\frac{\left(an^qX_n^{p+r}+bX_n^{q+r}\right)^{m+1}}{\left(cn^s-d\right)^m X_n^{m(s+1)}}\cdot\left(\sec\frac{\pi}{n}\right)^{m(s+1)}\cdot\frac{1}{n^{(m+1)(q+r-1)-ms}}\tag{2}$$

Proof: By *L.Fejes Tóth's* inequality (see e.g. [2])

$$\sum_{k=1}^{n}y_k=Y_n\leq\left(\cos\frac{\pi}{n}\right)\sum_{k=1}^{n}x_k=\left(\cos\frac{\pi}{n}\right)X_n,$$

and by (1) we deduce what we have to show. ■

Observation 2.1. If we put $p=q$, then by (2) we obtain:

$$\sum_{k=1}^{n}\frac{\left(aX_n^p+bx_k^p\right)^{m+1}x_k^{r(m+1)}}{\left(cY_n^s-dy_k^s\right)^m y_k^m}\geq\frac{\left(an^p+b\right)^{m+1}X_n^{(p+r)(m+1)}}{\left(cn^s-d\right)^m X_n^{m(s+1)}}\cdot\left(\sec\frac{\pi}{n}\right)^{m(s+1)}\cdot\frac{1}{n^{(m+1)(q+r-1)-ms}}=$$

$$=\frac{\left(an^p+b\right)^{m+1}X_n^{m(p+r-s-1)+p+r}}{\left(cn^s-d\right)^m n^{(m+1)(p+r-1)-ms}}\cdot\left(\sec\frac{\pi}{n}\right)^{m(s+1)}\tag{2'}$$

If we consider $p=s=0$, then by (2') we deduce that:

$$\sum_{k=1}^{n}\frac{x_k^{r(m+1)}}{y_k^m}\geq X_n^{m(r-1)+r}\cdot\frac{\left(\sec\frac{\pi}{n}\right)^m}{n^{(m+1)(r-1)}}\tag{2''}$$

If we take $r=1$ then by (2") yields:

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{y_k^m}\geq X_n\left(\sec\frac{\pi}{n}\right)^m\tag{2'''}$$

Remark 2.1. *Putting m =1 in (2"') we obtain the relation* (18) *from [2].*

Theorem 3. *If we have the notations presented above, then*

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{\sqrt[p]{(y_k y_{k+1}\cdots y_{k+p-1})^m}}\geq\frac{X_n^{m+1}}{p^{m-1}Y_n^m},\forall m\in R^+,p\in N^*-\{1\} \tag{3}$$

where $y_{n+j}=y_j,\forall j=\overline{0,p-1}$.

Proof: By AM-GM inequality we have:

$$\sqrt[p]{\prod_{j=0}^{p-1}y_{k+j}}\leq\frac{1}{p}\sum_{j=0}^{p-1}y_{k+j},\forall k=\overline{1,n},$$

which yields

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{\sqrt[p]{\left(\prod_{j=1}^{p-1}y_{k+j}\right)^m}}\geq p\sum_{k=1}^{n}\frac{x_k^{m+1}}{\left(\sum_{j=0}^{p-1}y_{k+j}\right)^m},$$

then by **(R)**

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{\sqrt[p]{\left(\prod_{j=1}^{p-1}y_{k+j}\right)^m}}\geq p\cdot\frac{X_n^{m+1}}{p^m Y_n^m}=\frac{X_n^{m+1}}{p^{m-1}Y_n^m},$$

which completes the proof. ■

Observation 3.1. *By* (3) *and L. Fejes Tóth's inequality we deduce that*:

$$\sum_{k=1}^{n}\frac{x_k^{m+1}}{\sqrt[p]{\left(\prod_{j=1}^{p-1}y_{k+j}\right)^m}}\geq\frac{X_n^{m+1}}{p^{m-1}X_n^m\left(\cos\frac{\pi}{2}\right)^m}=\frac{X_n}{p^{m-1}}\left(\sec\frac{\pi}{n}\right)^m \tag{3'}$$

which is a Radon-Erdös-Mordel type inequality.

If we consider m = 1, then by (3') we obtain that:

$$\sum_{k=1}^{n}\frac{x_k^2}{\sqrt[p]{\prod_{j=0}^{p-1}y_{k+j}}}\geq X_n\sec\frac{\pi}{n} \tag{3''}$$

For the same convex polygon $A_1A_2...A_n$ *(n≥3)*, $A_{n+1}\equiv A_1$ and for any point M of space which is not on the line A_kA_{k+1}, we denoted by $y_k(M)$ the distance from M to line A_kA_{k+1}, $s_k(M)=area\left[A_kMA_{k+1}\right],\ \forall k=\overline{1,n}$ and $S(M)=\sum_{k=1}^{n}s_k(M)$.

Theorem 4. *If M and N are the points in space which is not on the line A_kA_{k+1},* $\left(\forall k=\overline{1,n}\right)$ *, then:*

$$\sum_{k=1}^{n}a_k\left(\frac{1}{(y_k(M))^m}+\frac{1}{(y_k(N))^m}\right)\geq\frac{2p^{m+1}\left((S(M))^m+(S(N))^m\right)}{(S(M)S(N))^m} \tag{4}$$

Proof: We have:

$$U_n=\sum_{k=1}^{n}a_k\left(\frac{1}{(y_k(M))^m}+\frac{1}{(y_k(N))^m}\right)=\sum_{k=1}^{n}a_k^{m+1}\left(\frac{1}{(a_ky_k(M))^m}+\frac{1}{(a_ky_k(N))^m}\right)=$$

$$=\sum_{k=1}^{n}\frac{a_k^{m+1}}{2^m}\left(\frac{1}{S_k^m(M)}+\frac{1}{S_k^m(N)}\right),$$

where we apply **(R)** and yields

$$U_n\geq\frac{1}{2^m}\left(\frac{\left(\sum_{k=1}^{n}a_k\right)^{m+1}}{\left(\sum_{k=1}^{m}S_k(M)\right)^m}+\frac{\left(\sum_{k=1}^{n}a_k\right)^{m+1}}{\left(\sum_{k=1}^{n}S_k(N)\right)^m}\right)=$$

$$=\frac{(2p)^{m+1}}{2^m}\left(\frac{1}{(S(M))^m}+\frac{1}{(S(N))^n}\right)=\frac{2p^{m+1}\left((S(M))^m+(S(N))^m\right)}{(S(M)S(N))^m},$$

and this is what we had to prove. ■

Observation 4.1. *If M is a point inside of the polygon $A_1A_2...A_n$ and N is a point outside of the plane ($A_1A_2...A_n$), then the polygon is the base of the pyramid with the vertex N, so that* (4) *becomes*:

$$\sum_{k=1}^{n}a_k\left(\frac{1}{(y_k(M))^m}+\frac{1}{(y_k(N))^m}\right)\geq\frac{2p^{m+1}\left(S^m+S_l^m\right)}{(S\cdot S_l)^m}\qquad(4')$$

where S is the area of the polygon with the perimeter 2p, and S_l is the lateral area of the pyramid with vertex N and base the polygon $A_1A_2...A_n$.

If M and N are inside of the polygon then S(M) = S(N) = S, and (4') becomes:

$$\sum_{k=1}^{n}a_k\left(\frac{1}{(y_k(M))^m}+\frac{1}{(y_k(N))^m}\right)\geq\frac{4p^{m+1}S^m}{S^{2m}}=\frac{4p^{m+1}}{S^m}\qquad(4'')$$

If in addition M ≡ N, then we obtain that:

$$2\sum_{k=1}^{n}\frac{a_k}{(y_k(M))^m}\geq\frac{4p^{m+1}}{S^m}\Leftrightarrow\sum_{k=1}^{n}\frac{a_k}{(y_k(M))^m}\geq\frac{2p^{m+1}}{S^m}\qquad(4''')$$

Remark 4.1. *Taking m =1, then by (4''') we deduce that*:

$$\sum_{k=1}^{n}\frac{a_k}{y_k(M)}\geq\frac{2p^2}{S},$$

i.e. the problem 10876 *, proposed by D. Buşneag in G.M.-B nr. 1/1971, pp.35.*

For the convex polygon $A_1A_2...A_n$ $(n\geq3)$, $A_{n+1}\equiv A_1$ and *M* a point in space which is not on the line A_kA_{k+1}, $\left(\forall k=\overline{1,n}\right)$, we denote by $m_k=\mu\left(\angle A_kMA_{k+1}\right),k=\overline{1,n}$, the measure in radians of the angle $\angle A_kMA_{k+1}$, $k=\overline{1,n}$.

Theorem 5. *If $A_1A_2...A_n$ $(n\geq3)$ is a convex polygon as above, and M is a point in space which is not on the line A_kA_{k+1}, $\left(\forall k=\overline{1,n}\right)$, with $m_k\in\left(0,\frac{\pi}{2}\right],k=\overline{1,n}$, then:*

$$\sum_{k=1}^{n}\frac{a_k^2}{x_k x_{k+1}} \geq 4n\sin^2\frac{m_1+m_2+...+m_n}{2n} \tag{5}$$

Proof: In the triangle A_kMA_{k+1}, by the Law of Cosines we have,

$$a_k^2 = x_k^2 + x_{k+1}^2 - 2x_k x_{k+1}\cos(\angle A_k M A_{k+1}) = x_k^2 + x_{k+1}^2 - 2x_k x_{k+1}\cos m_k, \forall k=\overline{1,n},$$

where we apply AM-GM inequality and we obtain that:

$$a_k^2 \geq 2x_k x_{k+1}(1-\cos m_k) = 4x_k x_{k+1}\sin^2\frac{m_k}{2}, \forall k=\overline{1,n} \Leftrightarrow \frac{a_k^2}{x_k x_{k+1}} \geq 4\sin^2\frac{m_k}{2}, \forall k=\overline{1,n}.$$

Hence:

$$\sum_{k=1}^{n}\frac{a_k^2}{x_k x_{k+1}} \geq 4\sum_{k=1}^{n}\sin^2\frac{m_k}{2}.$$

Since the function $f:\left(0,\frac{\pi}{2}\right] \to R_+, f(x)=\sin^2\frac{x}{2}$ is convex on $\left(0,\frac{\pi}{2}\right]$, by Jensen's inequality we have that:

$$\sum_{k=1}^{n}\sin^2\frac{m_k}{2} \geq n\sin^2\frac{m_1+m_2+...+m_n}{2n}.$$

From the last two relations we get what must be demonstrated. ■

Remark 5.1. *If n≥5, $M \in IntA_1A_2...A_n$, then $\sum_{k=1}^{n} m_k = 2\pi$ and* (5) *becomes*:

$$\sum_{k=1}^{n}\frac{a_k^2}{x_k x_{k+1}} \geq 4n\sin^2\frac{\pi}{n},$$

*i.e. the problem*11386*, proposed by R.N. Gologan in G.M.-B nr. 8/1971, pp.487.*

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