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# THE PROBABILISTIC STABILITY FOR THE GAMMA FUNCTIONAL EQUATION

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***Abstract.*** *We obtain a stability result for the Baker functional equation, in the setting of probabilistic quasi-metric spaces. As a particular case, we discuss the probabilistic stability of the Gamma functional equation.*

***Keywords:*** *Hyers - Ulam stability, probabilistic quasi-metric space, probabilistic contraction.*

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## 1. INTRODUCTION

By using a fixed point technique, J. A. Baker [1] established the following Ulam - Hyers stability result for the nonlinear functional equation

$$f(x) = \Phi\left(x, f\left(\eta(x)\right)\right). \tag{1.1}$$

**Theorem 1.1 ([1], Theorem 2)** *Suppose S is a nonempty set, (X,d) is a complete metric space,* $\eta : S \to S$, $\Phi : S \times X \to X$, $\lambda \in [0,1)$, and

$$d\left(\Phi(u,x), \Phi(u,y)\right) \le \lambda d(x,y), \textit{ for all } u \in S,\ x, y \in X$$

*Also, suppose that f:S→X, δ>0, and*

$$d\left(f(u), \Phi\left(u, f\left(\eta(u)\right)\right)\right) \le \delta \textit{ for all } u \in S.$$

*Then there exists a unique mapping g:S→X such that*

$$g(u) = \Phi\left(u, g\left(\eta(u)\right)\right), \textit{ for all } u \in S,$$

*and*

$$d\left(f(u), g(u)\right) \le \frac{\delta}{1-\lambda}, \textit{ for all } u \in S.$$

The aim of this paper is to obtain a similar result in the setting of probabilistic quasi-metric spaces endowed with the łukasiewicz t-norm.

For the reader's convenience, we recall some useful terminology from the theory of probabilistic metric spaces. For more details, see the books [2] and [3].

A triangular norm (or t-norm) is a binary operation $T : [0,1] \times [0,1] \to [0,1]$ which is commutative, associative, monotone in each variable and has 1 as the unit element.

Some basic examples are

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$$T_L(a,\ b) = \max\{a+b-1,\ 0\} \text{ (the Lukasiewicz t-norm)}$$

$$T_P(a,\ b) = a \cdot b \text{ (the product t-norm)}$$

and

$$T_M(a,b) = \min\{a,b\} \text{ (the minimum t-norm).}$$

We denote by $\Delta_+$ the space of all functions $F:\mathbb{R}\to[0,1]$, such that $F$ is left-continuous and non-decreasing on $\mathbb{R}$, $F(0)=0$, and $F(\infty)=1$, and let $D_+$ be the subspace of $\Delta_+$ of functions $F$ with $\lim_{t\to\infty}F(t)=1$.

**Definition 1.1** *A probabilistic quasi-metric space is a triple (X,P,T), where X is a nonempty set, T is a t-norm, and $P:X\times X\to D_+$ is a mapping satisfying*

*(i)* $P_{xy} = P_{yx} = \varepsilon_0$ if and only if $x = y$;

*(ii)* $P_{xy}(t + s) \geq T(P_{xz}(t),\ Pzy(s)),\ x,\ y,\ z\in X,\ t,\ s>0$.

If $P$ has the additional symmetry property $P_{xy}=P_{yx}$ for all $x,y\in X$, then $(X,P,T)$ is called a Menger space.

If the mapping $P$ in Definition? has values in $\Delta_+$ instead of $D_+$, then $(X,P,T)$ is said to be a generalized probabilistic quasi-metric space.

The mapping $Q:X^2\to D_+$ defined by $Q_{xy} = P_{yx}$ for all $x,y\in X$ is called the conjugate probabilistic quasi-metric of $P$.

**Definition 1.2** *Let (X,P,T) be a probabilistic quasi-metric space. A sequence $(x_n)_n$ in X is said to be:*

*(i) right K-Cauchy (left K-Cauchy) if, for each $\varepsilon>0$ and $\lambda\in(0,1)$, there exists $k\in\mathbb{N}$ so that, for all $m\geq n\geq k$, $P_{x_n x_m}(\varepsilon)>1-\lambda$ $\left(Q_{x_n x_m}(\varepsilon)>1-\lambda \text{ respectively}\right)$;*

*(ii) P-convergent (Q-convergent) to $x\in X$ if, for each $\varepsilon>0$ and $\lambda\in(0,1)$, there exists $k\in\mathbb{N}$ so that $P_{xx_n}\left(\varepsilon>1-\lambda\left(Q_{xx_n}(\varepsilon>1-\lambda\right)\right.$, for all $n\geq k$.*

**Definition 1.3** *Let $A\in\{right\ K,\ left\ K\}$ and $B\in\{P,\ Q\}$. The space (X,P,T) is (A−B) complete if every A-Cauchy sequence is B convergent.*

**Definition 1.4** *The probabilistic quasi-metric space (X,P,T) has the L-US (R-US) property if every P- (Q-) convergent sequence has a unique limit.*

## 2. RESULTS

The proof of our main result is based on a fixed point theorem for $(\varepsilon-\lambda)$ - contractive mappings in probabilistic quasi-metric spaces (Lemma 2.1), which extends a result from [4]. Recall that an $(\varepsilon-\lambda)$ - contraction is a mapping $f$ from a Menger space $(X,F,T)$ to itself having the property that there exists $k\in(0,1)$ such that

$$\forall \varepsilon > 0,\ \lambda \in (0,1): F_{xy}(\varepsilon) > 1-\lambda \Rightarrow F_{f(x)f(y)}(k\varepsilon) > 1-k\lambda .$$

Note that every $(\varepsilon-\lambda)$- contraction satisfies

$$F_{f(x)f(y)}(k\varepsilon) > F_{xy}(t),\ \forall x,y \in X ,$$

that is, it is a Sehgal contraction on (*X*,*F*,*T*).

**Lemma 1** *Let (X, P, $T_L$ ) be a (right K−Q)-complete generalized probabilistic quasi-metric space with the R−US property, and let $f : X \to X$ be a mapping for which there exists $k \in (0,1)$ such that, for all $\varepsilon > 0$ and $\lambda \in (0,1)$,*

$$P_{xy}(\varepsilon) > 1-\lambda \Rightarrow P_{f(x)f(y)}(k\varepsilon) > 1-k\lambda \tag{2.1}$$

*Suppose there exist $\varepsilon > 0$, $\lambda \in (0,1)$ and $x \in X$ with $P_{xf(x)}(\varepsilon) > 1-\lambda$. Then the mapping f has a fixed point x*, and*

$$P_{xx^*}\left(\frac{\varepsilon}{1-k}+0\right) \geq \max\left\{1-\frac{\lambda}{1-k},0\right\} \tag{2.2}$$

*Proof:* Let $\varepsilon > 0$, $\lambda \in (0,1)$ and $x \in X$ be such that $P_{xf(x)}(\varepsilon) > 1-\lambda$. Inductively, we obtain that $P_{f^n(x)f^{n+1}(x)}(k^n\varepsilon) > 1-k^n\lambda$, for all $n \in \mathbb{N}$.

Let $t > 0$ and $\mu \in (0,1)$ be given. Since the series $\sum_{i=0}^{\infty} k^i$ is convergent, there exists $n_1 \in \mathbb{N}$ such that $\sum_{i=n_1}^{\infty} k^i\varepsilon < t$ and $\sum_{i=n_1}^{\infty} k^i\lambda < \mu$. Then, for all $n \geq n_1$ and $m \in \mathbb{N}^*$,

$$P_{f^n(x)f^{n+m}(x)}(t) \geq P_{f^n(x)f^{n+m}(x)}\left(\sum_{i=n}^{n+m-1} k^i\varepsilon\right)$$

$$\begin{aligned} &\geq (T_L)_{i=n}^{n+m-1}\left(P_{f^i(x)f^{i+1}(x)}(k^i\varepsilon)\right) \\ &\geq (T_L)_{i=n}^{n+m-1}\left(1-k^i\lambda\right) \\ &= \max\left\{1-\sum_{i=n}^{n+m-1} k^i\lambda,0\right\} > 1-\mu \end{aligned}$$

(2.3) **Error! Bookmark not defined.**

Consequently, $(f^n(x))_n$ is *right K*- Cauchy in *X*, thus it is *Q*-convergent to some $x^* \in X$, that is, $P_{f^n(x)x^*}(t) \to 1$ when $n \to \infty$, for all $t > 0$.

From hypothesis (2.1), we derive that *f* is a Sehgal contraction, with contraction constant *k*. Therefore

$$P_{f^{n+1}(x)f(x^*)}(kt) \geq P_{f^n(x)x^*}(t) \to 1,\ \forall t > 0 ,$$

meaning that $\left(f^n(x)\right)_n$ is *Q*-convergent to $f(x^*)$. By the *R−US* property of the space *X*, we conclude that $x^*$ is a fixed point of *f*.

Additionally, for all $n \geq 1$, relation (2.3.) implies

$$P_{xf^n(x)}\left(\sum_{i=0}^{n-1} k^i \varepsilon\right) \geq \max\left\{1 - \sum_{i=0}^{n-1} k^i \lambda, 0\right\}$$

so

$$P_{xf^n(x)}\left(\frac{\varepsilon}{1-k}\right) \geq P_{xf^n(x)}\left(\sum_{i=0}^{n-1} k^i \varepsilon\right) \geq \max\left\{1 - \frac{1-k^n}{1-k}\lambda, 0\right\}$$

$$\geq \max\left\{1 - \frac{\lambda}{1-k}, 0\right\}$$

For an arbitrary $\delta > 0$,

$$P_{xx^*}\left(\frac{\varepsilon}{1-k} + \delta\right) \geq T_L\left(P_{xf^n(x)}\left(\frac{\varepsilon}{1-k}\right), P_{f^n(x)x^*}(\delta)\right).$$

But $P_{f^n(x)x^*}(\delta) \to 1$ when $n \to \infty$. As a consequence,

$$P_{xx^*}\left(\frac{\varepsilon}{1-k} + \delta\right) \geq T_L\left(\max\left\{1 - \frac{\lambda}{1-k}, 0\right\}, 1\right) = \max\left\{1 - \frac{\lambda}{1-k}, 0\right\}$$

By letting $\delta \to 0$, we obtain the estimation (2.2). □

**Theorem 2.2** *Let S be a nonempty set, and* $(X, P, T_L)$ *be a (right K−Q) - complete generalized probabilistic quasi-metric space with the R−US property. Suppose that* $\Phi : S \times X \to X$ *is a mapping for which there exists* $k \in (0,1)$ *so that, for all* $\varepsilon > 0$ *and* $\lambda \in (0,1)$,

$$P_{xy}(\varepsilon) > 1 - \lambda \Rightarrow P\Phi(u,x)\Phi(u,y)(k\varepsilon) > 1 - k\lambda, \ \forall u \in S \tag{2.4}$$

*Then, for every f:S→X having the property that, for some* $\varepsilon > 0$ *and* $\lambda \in (0,1)$,

$$P_{f(u)\Phi(u,f(\eta(u)))}(\varepsilon) > 1 - \lambda, \ \forall u \in S \tag{2.5}$$

*there exists a mapping a:S→X satisfying the equation (1.1), with*

$$P_{f(u)a(u)}\left(\frac{\varepsilon}{1-k} + 0\right) \geq \max\left\{1 - \frac{\lambda}{1-k}, 0\right\}, \ \forall u \in S \tag{2.6}$$

*Proof:* We consider the space $Y = \{g : S \to X\}$ and Baker's operator $J : Y \to Y$ given by $J(g)(u) = \Phi(u, g(\eta(u)))$, for all $g \in Y$ and all $u \in S$. We define the mapping $F : Y \times Y \to D_+$ by

$$F_{gh}(t) = \sup_{s<\varepsilon} \inf_{u \in S} P_{g(u)h(u)}(s),$$

for all $g, h \in Y$. From the hypotheses on $(X, P, T_L)$, we infer that $(Y, F, T_L)$ is a (*right K−Q*) - complete generalized quasi-metric space with the *R−US* property.

Next, we show that, if $g,h \in Y$, and $\varepsilon > 0$ and $\lambda \in (0,1)$ are such that $F_{gh}(\varepsilon) > 1-\lambda$, then $F_{J(g)J(h)}(k\varepsilon) > 1-k\lambda$. To this end, first note that, if $F_{gh}(\varepsilon) > 1-\lambda$, there exists $\lambda' < \lambda$ in (0,1) for which $F_{gh}(\varepsilon) > 1-\lambda' > 1-\lambda$. This implies

$$\sup_{s<\varepsilon} \inf_{u\in S} P_{g(u)h(u)}(s_0) > 1-\lambda'$$

whence there exists $s_0 < \varepsilon$ with the property

$$\inf_{u\in S} P_{g(u)h(u)}(s_0) > 1-\lambda'.$$

It follows that

$$P_{g(u)h(u)}(s_0) > 1-\lambda',\ \forall u \in S$$

So

$$P_{g(\eta(u))h(\eta(u))}(s_0) > 1-\lambda',\ \forall u \in S$$

Then, via (2.4),

$$P_{J(g)(u)J(h)(u)}(ks_0) > 1-k\lambda',\ \forall u \in S$$

Therefore

$$F_{J(g)J(h)}(k\varepsilon) = \sup_{ks<k\varepsilon} \inf_{u\in S} P_{J(g)(u)J(h)(u)}(ks) \geq 1-k\lambda' > 1-k\lambda$$

Now, let *f* be a mapping satisfying (2.5), for some given $\varepsilon > 0$ and $\lambda \in (0,1)$. We claim that $F_{fJ(f)}(\varepsilon) > 1-\lambda$.

Indeed, from (2.5) it follows that there exists $\lambda' < \lambda$ with

$$P_{f(u)J(f)(u)}(\varepsilon) > 1-\lambda' \tag{2.7}$$

for all $u \in S$. By the left continuity of *P*, there exists $s_0 < \varepsilon$ with $P_{f(u)J(f)(u)}(s_0) > 1-\lambda'$, for all $u \in S$. We can deduce that $\inf_{u\in S} P_{f(u)J(f)(u)}(s_0) \geq 1-\lambda'$, so

$$F_{fJ(f)}(\varepsilon) \geq 1-\lambda' > 1-\lambda$$

One can now apply Lemma 2.1 to obtain that the operator *J* has a fixed point *a* in *Y*, meaning that the mapping $a: S \to X$ is an exact solution of (1.1). Moreover,

$$F_{fa}\left(\frac{\varepsilon}{1-k}+0\right) \geq \max\left\{1-\frac{\lambda}{1-k},0\right\},$$

providing the estimation (2.6). □

By setting $S=\mathbb{R}$, $X=\mathbb{R}$, $\Phi(u,x)=(u-1)x$ and $\eta(u)=u-1$ in the above theorem, we obtain the following probabilistic stability result for the Gamma functional equation:

**Theorem 2.3** *Let* (R, P, T$_L$) *be a (right K−Q) complete generalized probabilistic quasi-metric space with the R−US property. If there exists* $k \in (0,1)$ *so that, for all* $\varepsilon > 0$ *and* $\lambda \in (0,1)$,

$$P_{xy}(\varepsilon) > 1-\lambda \Rightarrow P_{(u-1)x,(u-1)y}(k\varepsilon) > 1-k\lambda,\ \forall u \in \mathbb{R},$$

*and* $f:\mathbb{R} \to \mathbb{R}$ *is a mapping satisfying*

$$P_{f(u),(u-1)f(u-1)}(\varepsilon) > 1-\lambda,\ \forall u \in \mathbb{R}$$

*for some* $\varepsilon > 0$ *and* $\lambda \in (0,1)$*, then there exists* $a : \mathbb{R} \to \mathbb{R}$ *with*

$$a(u) = (u-1)a(u-1),\ \forall u \in \mathbb{R}$$

*and*

$$P_{f(u)a(u)}\left(\frac{\varepsilon}{1-k} + 0\right) \ge \max\left\{1 - \frac{\lambda}{1-k}, 0\right\},\ \forall u \in \mathbb{R}.$$

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