***ORIGINAL PAPER***

# PROVING SOME GEOMETRIC IDENTITIES BY USING THE DETERMINANTS

DAM VAN NHI[1]

*Manuscript received: 10.07.2012; Accepted paper: 27.09.2012;*

*Published online: 01.12.2012.*

***Abstract.*** *In this note we will give proofs about some identities by using determinants. The method of determinants in Geometry is a powerful technique.*
***Keywords:*** *Matrix, determinant, tetrahedron, volume.*
***2000 Mathematical Subject Classification:*** *26D99, 97D50*

## 1. INTRODUCTION

Some results in geometry have been proved by the traditional methods. In this note, we want to use the identities to reprove and find out the new results. Successfully, we have gained the below achievements:

**Proposition 1.1.** Given the tetrahedron $A_1A_2A_3A_4$ with the lengths of 6 sides $a = l_{12}$; $b = l_{13}$; $c = l_{14}$; $x = l_{34}$; $y = l_{24}$; $z = l_{23}$. Put $2S = ax + by + cz$. Then, the radius of circumscribed sphere of the tetrahedron will be defined by the formula:

$$R = \frac{2\sqrt{2}\sqrt{S(S-ax)(S-by)(S-cz)}}{\sqrt{\begin{vmatrix} 2a^2 & a^2+b^2-z^2 & a^2+c^2-y^2 \\ a^2+b^2-z^2 & 2b^2 & b^2+c^2-x^2 \\ a^2+c^2-y^2 & b^2+c^2-x^2 & 2c^2 \end{vmatrix}}}$$

**Proposition 1.2.** Given the tetrahedron $ABCD$. Let $A_1$, $B_1$, $C_1$, $D_1$ be the barycenters of the surfaces $BCD$; $CDA$; $DAB$; $ABC$; correspondingly. Let's prove that with any point $O$ and choosing properly $u, v, t \in \{1, -1\}$ for

$$\left|V_{OB_1C_1D_1} + uV_{OC_1D_1A_1} + vV_{OD_1A_1B_1} + tV_{OA_1B_1C_1}\right| = \frac{1}{27}V_{ABCD}.$$

[1] Pedagogical University Ha Noi, Hanoi, Vietnam. Email: damvannhi@yahoo.com.

## 2. PROVING SOME IDENTITIES BY USING THE DETERMINANTS

**Lemma 2.1.** Decomposing the vectors $\vec{x}, \vec{y}, \vec{z}$ into the orthogonal basis, we have

$$\left(\left|\vec{x}, \vec{y}\right|.\vec{z}\right)^2 = \begin{vmatrix} \vec{x}.\vec{x} & \vec{x}.\vec{y} & \vec{x}.\vec{z} \\ \vec{y}.\vec{x} & \vec{y}.\vec{y} & \vec{y}.\vec{z} \\ \vec{z}.\vec{x} & \vec{z}.\vec{y} & \vec{z}.\vec{z} \end{vmatrix}$$

*Proof:* Suppose that $\vec{x} = (a,b,c)$, $\vec{y} = (a',b',c')$ and $\vec{z} = (a'',b'',c'')$. Since

$$\left(\left|\vec{x}, \vec{y}\right|.\vec{z}\right)^2 = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}^2 = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}\begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix}$$

therefore we have $\left(\left|\vec{x}, \vec{y}\right|.\vec{z}\right)^2 = \begin{vmatrix} \vec{x}.\vec{x} & \vec{x}.\vec{y} & \vec{x}.\vec{z} \\ \vec{y}.\vec{x} & \vec{y}.\vec{y} & \vec{y}.\vec{z} \\ \vec{x}.\vec{z} & \vec{y}.\vec{z} & \vec{z}.\vec{z} \end{vmatrix}$.

**Proposition 2.2.** The volume of a triangular pyramid *SABC* with *SA* = *a*, *SB* = *b*, *SC* = *c*, *BC* = *x*, *CA* = *y*, *AB* = *z* is

$$V = \frac{1}{12\sqrt{2}}\sqrt{\begin{vmatrix} 2a^2 & a^2+b^2-z^2 & a^2+c^2-y^2 \\ a^2+b^2-z^2 & 2b^2 & b^2+c^2-x^2 \\ a^2+c^2-y^2 & b^2+c^2-x^2 & 2c^2 \end{vmatrix}}.$$

*Proof.* Denote $\vec{x} = \overrightarrow{SA}$, $\vec{y} = \overrightarrow{SB}$, $\vec{z} = \overrightarrow{SC}$ and $V = V_{SABC}$. We have $\vec{x}.\vec{y} = a^2+b^2-z^2$, $\vec{x}.\vec{z} = a^2+c^2-y^2$ and $\vec{y}.\vec{z} = b^2+c^2-x^2$. Using $36V^2 = \begin{vmatrix} \vec{x}.\vec{x} & \vec{x}.\vec{y} & \vec{x}.\vec{z} \\ \vec{y}.\vec{x} & \vec{y}.\vec{y} & \vec{y}.\vec{z} \\ \vec{x}.\vec{z} & \vec{y}.\vec{z} & \vec{z}.\vec{z} \end{vmatrix}$ by Lemma 2.1 we have $288V^2 = \begin{vmatrix} 2a^2 & a^2+b^2-z^2 & a^2+c^2-y^2 \\ a^2+b^2-z^2 & 2b^2 & b^2+c^2-x^2 \\ a^2+c^2-y^2 & b^2+c^2-x^2 & 2c^2 \end{vmatrix}$.

**Lemma 2.3. [2]** Given the tetrahedron *ABCD* with $A_1(x_1, y_1, z_1)$, $A_2(x_2, y_2, z_2)$, $A_3(x_3, y_3, z_3)$ and $A_4(x_4, y_4, z_4)$. Denoting *R* is the radius of circumscribed sphere of the tetrahedron *ABCD* and $l_{ij} = l_{ji}$ is the length of side $A_iA_j = A_jA_i$, $i \neq j$. Then, we always have

$$R = \frac{1}{24V}\sqrt{2l_{13}^2l_{14}^2l_{23}^2l_{24}^2 + 2l_{12}^2l_{14}^2l_{32}^2l_{34}^2 + 2l_{12}^2l_{13}^2l_{42}^2l_{43}^2 - l_{12}^4l_{34}^4 - l_{13}^4l_{24}^4 - l_{14}^4l_{23}^4}.$$

*Proof:* It was well-known that the volume of a tetrahedron is not changed under a translation. Therefore, without loss of generality we can suppose that the center of the circumscribed sphere of the tetrahedron $A_1A_2A_3A_4$ is the orgin of coordinates. Then we have

$$\begin{cases} x_1^2 + y_1^2 + z_1^2 - R^2 = 0 \\ x_2^2 + y_2^2 + z_2^2 - R^2 = 0 \\ x_3^2 + y_3^2 + z_3^2 - R^2 = 0 \\ x_4^2 + y_4^2 + z_4^2 - R^2 = 0 \end{cases}.$$

From this, we deduce that $R^2 = \dfrac{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}{2} + x_i x_j + y_i y_j + z_i z_j$ with any $i, j = 1, 2, 3, 4, i \neq j$. Let's $t_{ij} = R^2 - x_i x_j - y_i y_j - z_i z_j$. Then $t_{ij} = t_{ji} = \dfrac{(x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2}{2} = \dfrac{l_{ij}^2}{2} = \dfrac{l_{ji}^2}{2}$. Consider the following decomposition

$$P = \begin{vmatrix} x_1 & y_1 & z_1 & R \\ x_2 & y_2 & z_2 & R \\ x_3 & y_3 & z_3 & R \\ x_4 & y_4 & z_4 & R \end{vmatrix} \cdot \begin{vmatrix} -x_1 & -y_1 & -z_1 & R \\ -x_2 & -y_2 & -z_2 & R \\ -x_3 & -y_3 & -z_3 & R \\ -x_4 & -y_4 & -z_4 & R \end{vmatrix} =$$

$$= \begin{vmatrix} x_1 & y_1 & z_1 & R \\ x_2 & y_2 & z_2 & R \\ x_3 & y_3 & z_3 & R \\ x_4 & y_4 & z_4 & R \end{vmatrix} \cdot \begin{vmatrix} -x_1 & -x_2 & -x_3 & -x_4 \\ -y_1 & -y_2 & -y_3 & -y_4 \\ -z_1 & -z_2 & -z_3 & -z_4 \\ R & R & R & R \end{vmatrix} = \begin{vmatrix} 0 & t_{12} & t_{13} & t_{14} \\ t_{21} & 0 & t_{23} & t_{24} \\ t_{31} & t_{32} & 0 & t_{34} \\ t_{41} & t_{42} & t_{43} & 0 \end{vmatrix}$$

with $t_{ij} = t_{ji}$. Because $\left\| \begin{matrix} x_1 & y_1 & z_1 & R \\ x_2 & y_2 & z_2 & R \\ x_3 & y_3 & z_3 & R \\ x_4 & y_4 & z_4 & R \end{matrix} \right\|$ and $\left\| \begin{matrix} -x_1 & -y_1 & -z_1 & R \\ -x_2 & -y_2 & -z_2 & R \\ -x_3 & -y_3 & -z_3 & R \\ -x_4 & -y_4 & -z_4 & R \end{matrix} \right\|$ all equals with $6RV$ so

$$36R^2V^2 = \left\| \begin{matrix} 0 & \frac{l_{12}^2}{2} & \frac{l_{13}^2}{2} & \frac{l_{14}^2}{2} \\ \frac{l_{21}^2}{2} & 0 & \frac{l_{23}^2}{2} & \frac{l_{24}^2}{2} \\ \frac{l_{31}^2}{2} & \frac{l_{32}^2}{2} & 0 & \frac{l_{34}^2}{2} \\ \frac{l_{41}^2}{2} & \frac{l_{42}^2}{2} & \frac{l_{43}^2}{2} & 0 \end{matrix} \right\| = \frac{1}{16} \left\| \begin{matrix} 0 & l_{12}^2 & l_{13}^2 & l_{14}^2 \\ l_{21}^2 & 0 & l_{23}^2 & l_{24}^2 \\ l_{31}^2 & l_{32}^2 & 0 & l_{34}^2 \\ l_{41}^2 & l_{42}^2 & l_{43}^2 & 0 \end{matrix} \right\|$$

From the last determinant and taking the root, we have the formula calculating

$$R = \frac{1}{24V}\sqrt{2l_{13}^2 l_{14}^2 l_{23}^2 l_{24}^2 + 2l_{12}^2 l_{14}^2 l_{32}^2 l_{34}^2 + 2l_{12}^2 l_{13}^2 l_{42}^2 l_{43}^2 - l_{12}^4 l_{34}^4 - l_{13}^4 l_{24}^4 - l_{14}^4 l_{23}^4} \qquad \square$$

From the Proposition 2.2 and the Lemma 2.3 we can deduce the formula calculating the radius of circumscribed sphere through the lengths of 6 sides of the quadrilateral.

**Proposition 2.4.** Given the tetrahedron $A_1A_2A_3A_4$ with the lengths of 6 sides $a = l_{12}$, $b = l_{13}$, $c = l_{14}$, $x = l_{34}$, $y = l_{24}$, $z = l_{23}$. Put $2S = ax + by + cz$. Then, the radius of circumscribed sphere of the tetrahedron will be defined by the formula:

$$R = \frac{2\sqrt{2}\sqrt{S(S-ax)(S-by)(S-cz)}}{\sqrt{\begin{vmatrix} 2a^2 & a^2+b^2-z^2 & a^2+c^2-y^2 \\ a^2+b^2-z^2 & 2b^2 & b^2+c^2-x^2 \\ a^2+c^2-y^2 & b^2+c^2-x^2 & 2c^2 \end{vmatrix}}}.$$

*Proof:* By the Ptolemy's Inequality we have

$$ax + by \geq cz,$$

$$by + cz \geq ax$$

and

$$cz + ax \geq by.$$

We can easily check the formula

$$16S(S-ax)(S-by)(S-cz) =$$
$$= 2l_{13}^2 l_{14}^2 l_{23}^2 l_{24}^2 + 2l_{12}^2 l_{14}^2 l_{32}^2 l_{34}^2 + 2l_{12}^2 l_{13}^2 l_{42}^2 l_{43}^2 - l_{12}^4 l_{34}^4 - l_{13}^4 l_{24}^4 - l_{14}^4 l_{23}^4$$

Therefore we have $R = \frac{\sqrt{S(S-ax)(S-by)(S-cz)}}{6V}$. Hence, we obtain the relationship $R = \frac{2\sqrt{2}\sqrt{S(S-ax)(S-by)(S-cz)}}{\sqrt{\begin{vmatrix} 2a^2 & a^2+b^2-z^2 & a^2+c^2-y^2 \\ a^2+b^2-z^2 & 2b^2 & b^2+c^2-x^2 \\ a^2+c^2-y^2 & b^2+c^2-x^2 & 2c^2 \end{vmatrix}}}$. □

**Example.** With the tetrahedron $A_1A_2A_3A_4$. Let's prove that with any points O we always have the inequality:

$$V_{A_1A_2A_3A_4} \leq \frac{1}{6}\prod_{i=1}^{4}\sqrt{OA_i^2 + 1}.$$

*Solution:* Building the coordinates $Oxyz$. Suppose $A_i(x_i, y_i, z_i)$, $i$ = 1, 2, 3, 4. We have

$$36V_{A_1A_2A_3A_4}^2 = \begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}^2 \leq \prod_{i=1}^{4}\left(x_i^2 + y_i^2 + z_i^2 + 1\right)$$

following the Hadamard's Inequality. Therefore $V_{A_1A_2A_3A_4} \leq \frac{1}{6}\prod_{i=1}^{4}\sqrt{OA_i^2 + 1}$. □

**Example.** Given the trihedral angle $Sxyz$. Suppose that the point $N$ is in the trihedron and the plane *(P)* through $N$ cuts $Sx$, $Sy$, $Sz$ at $A$, $B$, $C$, correspondingly. Then the volume ratio $\frac{V_{SABC}^2}{V_{SNBC}.V_{SNCA}.V_{SNAB}}$ is constant.

*Solution:* Calling the distance from $N$ to *(SBC)*, *(SCA)*, *(SAB)* is $a$, $b$, $c$, correspondingly and setting $SA=x$, $SB=y$, $SC=z$. Then $a$, $b$, $c$ is constant and we have

$$\frac{V_{SABC}^2}{V_{SNBC}.V_{SNCA}.V_{SNAB}} = \frac{6^3 V_{SABC}^2}{(ayz).(bzx).(cxy)} = \frac{6^3 V_{SABC}^2}{abcx^2y^2z^2}.$$

On the other hand, we also have

$$6^2 V_{SABC}^2 = x^2 y^2 z^2 \left(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma\right)$$

following the Lemma 2.1, in which $\alpha = \angle BSC$, $\beta = \angle CSA$, $\gamma = \angle ASB$. Therefore, the volume ratio which is calculated by

$$\frac{V_{SABC}^2}{V_{SNBC}.V_{SNCA}.V_{SNAB}} = \frac{6\left(1 - \cos^2\alpha - \cos^2\beta - \cos^2\gamma + 2\cos\alpha\cos\beta\cos\gamma\right)}{abc}$$

is constant.

**Example.** Given the point *O* in the tetrahedron *ABCD*. A plane *(P)* through *O* cuts *AB,AC,AD*. Signing the distance from *A, B, C, D* to *(P)* is $t_a, t_b, t_c, t_d$, correspondingly. Prove that $t_a V_{OBCD} = t_b V_{OCDA} + t_c V_{ODAB} + t_d V_{OABC}$.

*Solution:* Building the coordinates *Oxyz* so that *O(0, 0, 0), (P) : z = 0* and $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$, $D(x_4, y_4, z_4)$ with $t_a = z_1 > 0,\ t_b = -z_2 > 0;\ t_c = -z_3 > 0;\ t_d = -z_4 > 0$. Conventing that the volume of the tetrahedron is calculated by the determinants following the right hand rule:

$$t_a V_{OBCD} = z_1 \begin{vmatrix} 0 & 0 & 0 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} = -z_1 \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}$$

$$t_b V_{OCDA} = -z_2 \begin{vmatrix} 0 & 0 & 0 & 1 \\ x_4 & y_4 & z_4 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_1 & y_1 & z_1 & 1 \end{vmatrix} = -z_2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix}$$

$$t_c V_{ODAB} = -z_3 \begin{vmatrix} 0 & 0 & 0 & 1 \\ x_4 & y_4 & z_4 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = z_3 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix}$$

$$t_d V_{OABC} = -z_4 \begin{vmatrix} 0 & 0 & 0 & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_2 & y_2 & z_2 & 1 \end{vmatrix} = -z_4 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}$$

Therefore, the difference $T = t_a V_{OBCD} - t_b V_{OCDA} - t_c V_{ODAB} - t_d V_{OABC}$ exactly equals

$$T = -z_1 \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} + z_2 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} - z_3 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_4 & y_4 & z_4 \end{vmatrix} + z_4 \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 & z_1 & z_1 \\ x_2 & y_2 & z_2 & z_2 \\ x_3 & y_3 & z_3 & z_3 \\ x_4 & y_4 & z_4 & z_4 \end{vmatrix} = 0.$$

Summarizing $t_a V_{OBCD} = t_b V_{OCDA} + t_c V_{ODAB} + t_d V_{OABC}$. □

**Proposition 2.5.** Given the tetrahedron *ABCD*. Let $A_1, B_1, C_1, D_1$ be the barycenters of the surfaces *BCD, CDA, DAB, ABC*, correspondingly. Let's prove that with any point *O* and choosing properly $u$, $v$, $t \in \{1,-1\}$ for

$$\left|V_{OB_1C_1D_1} + uV_{OC_1D_1A_1} + vV_{OD_1A_1B_1} + V_{OA_1B_1C_1}\right| = \frac{1}{27}V_{ABCD}.$$

*Proof:* Builiding the coordinates *Oxyz* so that *O(0, 0, 0)*, $A(x_1, y_1, z_1)$, $B(x_2, y_2, z_2)$, $C(x_3, y_3, z_3)$ and $D(x_4, y_4, z_4)$. The coordinates of the barycenters of the surfaces of the tetrahedron *ABCD*

$$A_1\left(\frac{x_2+x_3+x_4}{3}, \frac{y_2+y_3+y_4}{3}, \frac{z_2+z_3+z_4}{3}\right)$$
$$B_1\left(\frac{x_1+x_3+x_4}{3}, \frac{y_1+y_3+y_4}{3}, \frac{z_1+z_3+z_4}{3}\right)$$
$$C_1\left(\frac{x_2+x_1+x_4}{3}, \frac{y_2+y_1+y_4}{3}, \frac{z_2+z_1+z_4}{3}\right)$$
$$D_1\left(\frac{x_2+x_3+x_1}{3}, \frac{y_2+y_3+y_1}{3}, \frac{z_2+z_3+z_1}{3}\right)$$

Let's $X = \sum_{i=1}^{4} x_i$, $Y = \sum_{i=1}^{4} y_i$ and $Z = \sum_{i=1}^{4} z_i$, setting $V_a = 6V_{OB_1C_1D_1}$. We perform

$$V_a = \pm\begin{vmatrix} 0 & 0 & 0 & 1 \\ \frac{x_1+x_3+x_4}{3} & \frac{y_1+y_3+y_4}{3} & \frac{z_1+z_3+z_4}{3} & 1 \\ \frac{x_2+x_1+x_4}{3} & \frac{y_2+y_1+y_4}{3} & \frac{z_2+z_1+z_4}{3} & 1 \\ \frac{x_2+x_3+x_1}{3} & \frac{y_2+y_3+y_1}{3} & \frac{z_2+z_3+z_1}{3} & 1 \end{vmatrix} =$$

$$= \frac{\pm 1}{27}\begin{vmatrix} X-x_2 & Y-y_2 & Z-z_2 \\ X-x_3 & Y-y_3 & Z-z_3 \\ X-x_4 & Y-y_4 & Z-z_4 \end{vmatrix} = \frac{\pm 1}{27}\begin{vmatrix} X & Y & Z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

Signal

$$T = \begin{vmatrix} X & Y & Z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \pm \begin{vmatrix} X & Y & Z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \pm \begin{vmatrix} X & Y & Z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} \pm \begin{vmatrix} X & Y & Z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix}$$

with the proper choice of +, - . From $\begin{vmatrix} X & Y & Z & 1 & 1 \\ x_1 & y_1 & z_1 & 1 & 1 \\ x_2 & y_2 & z_2 & 1 & 1 \\ x_3 & y_3 & z_3 & 1 & 1 \\ x_4 & y_4 & z_4 & 1 & 1 \end{vmatrix} = 0$ deducing

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} - \begin{vmatrix} X & Y & Z & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} + \begin{vmatrix} X & Y & Z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} - \begin{vmatrix} X & Y & Z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix} + \begin{vmatrix} X & Y & Z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0 .$$

Therefore $\pm V_{ABCD} \pm 27V_{OB_1C_1D_1} \pm 27V_{OC_1D_1A_1} \pm 27V_{OD_1A_1B_1} \pm 27V_{OA_1B_1C_1} = 0$. From this, we deduce the result: Existing a proper choice $u,\ v,\ t\ \in\ \{1,-1\}$ so that

$$\left|V_{OB_1C_1D_1} + uV_{OC_1D_1A_1} + vV_{OD_1A_1B_1} + V_{OA_1B_1C_1}\right| = \frac{1}{27}V_{ABCD}. \qquad \square$$

## 3. SOME EXAMPLES ABOUT AREA

**Proposition 3.1.** Given *ΔABC*. Taking the points *M, N, P* belonging to the edges *BC, CA, AB*, correspondingly. Let $u = \frac{MB}{MC}$, $v = \frac{NC}{NA}$, $t = \frac{PA}{PB}$. Then we have the area ratio

$$\frac{S_{MNP}}{S_{ABC}} = \frac{1+uvt}{(1+u)(1+v)(1+t)}.$$

*Proof:* Suppose $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$. Because $\overrightarrow{BM} = u\overrightarrow{MC}$,

$$M\left(\frac{x_2+ux_3}{1+u}, \frac{y_2+uy_3}{1+u}\right).$$

Similarly, we also have $N\left(\frac{x_3+vx_1}{1+v}, \frac{y_3+vy_1}{1+v}\right)$ and $P\left(\frac{x_1+tx_2}{1+t}, \frac{y_1+ty_2}{1+t}\right)$. Let $\alpha = (1+u)(1+v)(1+t)$ and change

$$T = \begin{vmatrix} 1 & 1 & 1 \\ \frac{x_2+ux_3}{1+u} & \frac{x_3+vx_1}{1+v} & \frac{x_1+tx_2}{1+t} \\ \frac{y_2+uy_3}{1+u} & \frac{y_3+vy_1}{1+v} & \frac{y_1+ty_2}{1+t} \end{vmatrix} =$$

$$= \frac{1}{(1+u)(1+v)(1+t)} \begin{vmatrix} 1+u & 1+v & 1+t \\ x_2+ux_3 & x_3+vx_1 & x_1+tx_2 \\ y_2+uy_3 & y_3+vy_1 & y_1+ty_2 \end{vmatrix} =$$

$$= \frac{1}{\alpha} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \begin{vmatrix} 0 & v & 1 \\ 1 & 0 & t \\ u & 1 & 0 \end{vmatrix} = \frac{1+uvt}{\alpha} \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}.$$

Therefore $\frac{S_{MNP}}{S_{ABC}} = \frac{gtt\begin{vmatrix} 1 & 1 & 1 \\ \frac{x_2+ux_3}{1+u} & \frac{x_3+vx_1}{1+v} & \frac{x_1+tx_2}{1+t} \\ \frac{y_2+uy_3}{1+u} & \frac{y_3+vy_1}{1+v} & \frac{y_1+ty_2}{1+t} \end{vmatrix}}{gtt\begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix}} = \frac{1+uvt}{(1+u)(1+v)(1+t)}$

or $\frac{S_{MNP}}{S_{ABC}} = \frac{1+uvt}{(1+u)(1+v)(1+t)}$.

**Example. [The VN 6th Olympic Mathematics]** Given *ΔABC* with the area equals to 1. Taking the points *M, N, P* belonging to the edges *BC, CA, AB*, correspondingly so that $k_1 = \frac{MB}{MC}$, $k_2 = \frac{NC}{NA}$, $k_3 = \frac{PA}{PB}$ with $k_1, k_2, k_3 < 1$. Determining the area of the triangle with the vertices are the intersections among *AM, BN, CP*.

*Solution:* Suppose $A(x_1, y_1), B(x_2, y_2), C(x_3, y_3)$. Because $\overrightarrow{BM} = k_1\overrightarrow{MC}$,

$$M\left(x_m = \frac{x_2+k_1x_3}{1+k_1},\ y_m = \frac{y_2+k_1y_3}{1+k_1}\right).$$

Similarly, we also have

$$N\left(x_n = \frac{x_3+k_2x_1}{1+k_2},\ y_n = \frac{y_3+k_2y_1}{1+k_2}\right)$$

and

$$P\left(x_p = \frac{x_1+k_3x_2}{1+k_3},\ y_p = \frac{y_1+k_3y_2}{1+k_3}\right).$$

Lets $E = AM \times BN$. Following the Menlaut's theorem, we have $\frac{AE}{EM}\frac{BM}{BC}\frac{NC}{NA} = 1$. So $AE = \frac{1+k_1}{k_1k_2}EM$ and $E\left(x_e = \frac{k_1k_2x_1+(1+k_1)x_m}{1+k_1+k_1k_2}, y_e = \frac{k_1k_2y_1+(1+k_1)y_m}{1+k_1+k_1k_2}\right)$. From this, we deduce that

$$E\left(x_e = \frac{k_1k_2x_1+x_2+k_1x_3}{1+k_1+k_1k_2}, y_e = \frac{k_1k_2y_1+y_2+k_1y_3}{1+k_1+k_1k_2}\right)$$

Let's $F = BN \times CP$, $G = CP \times AM$. Similarly as above, we receive

$$F\left(x_f = \frac{k_2k_3x_2+x_4+k_2x_1}{1+k_2+k_2k_3}, y_f = \frac{k_2k_3y_2+y_2+k_2y_1}{1+k_2+k_2k_3}\right)$$

$$G\left(x_g = \frac{k_3k_1x_3+x_1+k_3x_2}{1+k_3+k_3k_1}, y_e = \frac{k_3k_1y_3+y_1+k_3y_2}{1+k_3+k_3k_1}\right).$$

Setting $\delta = (1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)$ and implementing the changes

$$T=\begin{vmatrix}1 & 1 & 1\\ \frac{k_1k_2x_1+x_2+k_1x_3}{1+k_1+k_1k_2} & \frac{k_2k_3x_2+x_4+k_2x_1}{1+k_2+k_2k_3} & \frac{k_3k_1x_3+x_1+k_3x_2}{1+k_3+k_3k_1}\\ \frac{k_1k_2y_1+y_2+k_1y_3}{1+k_1+k_1k_2} & \frac{k_2k_3y_2+y_2+k_2y_1}{1+k_2+k_2k_3} & \frac{k_3k_1y_3+y_1+k_3y_2}{1+k_3+k_3k_1}\end{vmatrix}=$$

$$=\frac{1}{\delta}\begin{vmatrix}1+k_1+k_1k_2 & 1+k_2+k_2k_3 & 1+k_3+k_3k_1\\ k_1k_2x_1+x_2+k_1x_3 & k_2k_3x_2+x_4+k_2x_1 & k_3k_1y_3+y_1+k_3y_2\\ k_1k_2y_1+y_2+k_1y_3 & k_2k_3y_2+y_2+k_2y_1 & k_3k_1y_3+y_1+k_3y_2\end{vmatrix}=$$

$$=\frac{1}{\delta}\begin{vmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{vmatrix}\begin{vmatrix}k_1k_2 & k_2 & 1\\ 1 & k_2k_3 & k_3\\ k_1 & 1 & k_3k_1\end{vmatrix}=\frac{(1-k_1k_2k_3)^2}{\delta}\begin{vmatrix}1&1&1\\x_1&x_2&x_3\\y_1&y_2&y_3\end{vmatrix}.$$

From this we deduce $S_{EFG}=\dfrac{(1-k_1k_2k_3)^2}{(1+k_1+k_1k_2)(1+k_2+k_2k_3)(1+k_3+k_3k_1)}$. □

**Example.** Given *ΔABC* is acute triangle with the radiuses of the circumscribed and inscribed circles *R, r*. Suppose that the roots of the hights are $A_0$, $B_0$, $C_0$. Prove that

$$\frac{S_{A_0B_0C_0}}{S_{ABC}}\le\frac{2}{27}\left(1+\frac{r}{R}\right)^3.$$

*Solution:* Because $\dfrac{A_0B}{A_0C}=\dfrac{c\cos B}{b\cos C}$, $\dfrac{B_0C}{B_0A}=\dfrac{a\cos C}{c\cos A}$ and $\dfrac{C_0A}{C_0B}=\dfrac{b\cos A}{a\cos B}$,

$$\frac{S_{A_0B_0C_0}}{S_{ABC}}=\frac{2abc\cos A\cos B\cos C}{abc}=2\cos A\cos B\cos C.$$

Because of $1+\dfrac{r}{R}=\cos A+\cos B+\cos C$ we have $\dfrac{S_{A_0B_0C_0}}{S_{ABC}}\le\dfrac{2}{27}\left(1+\dfrac{r}{R}\right)^3$ following the Cauchy's Inequality. □

**Example.** Given the triangle *ABC*. Let *M, N, P* be the midpoints of *BC, CA* and *AB*. Prove that with any point *O* outside the *ΔABC* having one of the areas $S_{OAM}$, $S_{OBN}$, $S_{OCP}$ equals the sum of the two remaining areas.

*Solution:* Building the coordinates *Oxy* so that *O(0, 0)* and $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$. The coordinates of the midpoints $M\left(\dfrac{x_2+x_3}{2},\dfrac{y_2+y_3}{2}\right)$, $N\left(\dfrac{x_3+x_1}{2},\dfrac{y_3+y_1}{2}\right)$, $P\left(\dfrac{x_1+x_2}{2},\dfrac{y_1+y_2}{2}\right)$. Let's consider the case *O* belonging to $\angle AGN$, (*G* is the forcus of *ΔABC*). Conventing the area of triangle is calculated by the determinants following the right hand rule, we have:

$$2S_{OAM} = 2\begin{vmatrix} 0 & 0 & 1 \\ x_1 & y_1 & 1 \\ \frac{x_2+x_3}{2} & \frac{y_2+y_3}{2} & 1 \end{vmatrix} = \begin{vmatrix} x_1 & y_1 \\ x_2+x_3 & y_2+y_3 \end{vmatrix}$$

$$2S_{OBN} = 2\begin{vmatrix} 0 & 0 & 1 \\ x_2 & y_2 & 1 \\ \frac{x_3+x_1}{2} & \frac{y_3+y_1}{2} & 1 \end{vmatrix} = \begin{vmatrix} x_2 & y_2 \\ x_3+x_1 & y_3+y_1 \end{vmatrix}$$

$$2S_{OCP} = 2\begin{vmatrix} 0 & 0 & 1 \\ \frac{x_1+x_2}{2} & \frac{y_1+y_2}{2} & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = \begin{vmatrix} x_1+x_2 & y_1+y_2 \\ x_3 & y_3 \end{vmatrix}$$. Nh vy

$2S_{OAM} + 2S_{OBN} = \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_1 & y_1 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} = 2S_{OCP}$ because all are equal to $\begin{vmatrix} x_1 & y_1 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix}$. Similarly for the other situations. □

## REFERENCES


[1] Pogorelov, A., *Geometry*, Mir Publishers, Moscow, 1987.
[2] Proskuryakov, I. V., *Sbornik zadatr po liheinoi algebre*, Nauka, 1978.