ORIGINAL PAPER

CHARACTERIZATION OF PARALLEL SURFACES TO $\mathsf{S}-\alpha$ SURFACES IN HEISENBERG GROUP HEIS^3

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Abstract. *In this paper, we study parallel surfaces to $\mathsf{S}-\alpha$ surfaces according to Sabban frame in the Heisenberg group Heis^3.*

Keywords: *Biharmonic curve, Heisenberg group, Sabban frame.*

Mathematics Subject Classifications: *53C41, 53A10.*

1. INTRODUCTION

A smooth map $\phi : N \to M$ is said to be biharmonic if it is a critical point of the bienergy functional:

$$E_2(\phi) = \int_N \frac{1}{2} \left| \mathsf{T}(\phi) \right|^2 dv_h,$$

where $\mathsf{T}(\phi) := \operatorname{tr}\nabla^{\phi} d\phi$ is the tension field of ϕ

The Euler – Lagrange equation of the bienergy is given by $\mathsf{T}_2(\phi) = 0$. Here the section $\mathsf{T}_2(\phi)$ is defined by

$$\mathsf{T}_2(\phi) = -\Delta_{\phi}\mathsf{T}(\phi) + \operatorname{tr}R(\mathsf{T}(\phi), d\phi)d\phi, \tag{1.1}$$

and called the bitension field of ϕ. Non-harmonic biharmonic maps are called proper biharmonic maps.

In this paper, we study parallel surfaces to $\mathsf{S}-\alpha$ surfaces according to Sabban frame in the Heisenberg group Heis^3. We characterize the biharmonic curves in terms of their geodesic curvature in the Heisenberg group Heis^3. Finally, we find explicit parametric equations of parallel surfaces to $\mathsf{S}-\alpha$ surfaces according to Sabban Frame.

2. BACKROUND ON BIHARMONIC S-CURVES ACCORDING TO SABBAN FRAME IN THE HEISENBERG GROUP HEIS^3

The Riemannian metric g on Heis^3 is given by

$$g = dx^2 + dy^2 + (dz - xdy)^2. \tag{2.1}$$

The Lie algebra of Heis^3 has an orthonormal basis

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$$\mathbf{e}_1 = \frac{\partial}{\partial x}, \mathbf{e}_2 = \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \mathbf{e}_3 = \frac{\partial}{\partial z}, \tag{2.2}$$

for which we have the Lie products

$$[\mathbf{e}_1, \mathbf{e}_2] = \mathbf{e}_3, [\mathbf{e}_2, \mathbf{e}_3] = [\mathbf{e}_3, \mathbf{e}_1] = 0$$

with

$$g(\mathbf{e}_1, \mathbf{e}_1) = g(\mathbf{e}_2, \mathbf{e}_2) = g(\mathbf{e}_3, \mathbf{e}_3) = 1.$$

Let $\gamma : I \to Heis^3$ be a non geodesic curve on the Heisenberg group Heis^3 parametrized by arc length. Let $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ be the Frenet frame fields tangent to the Heisenberg group Heis^3 along γ defined as follows:

$\mathbf{T}$ is the unit vector field γ' tangent to γ, $\mathbf{N}$ is the unit vector field in the direction of $\nabla_{\mathbf{T}}\mathbf{T}$ (normal to γ), and $\mathbf{B}$ is chosen so that $\{\mathbf{T}, \mathbf{N}, \mathbf{B}\}$ is a positively oriented orthonormal basis. Then, we have the following Frenet formulas:

$$\nabla_{\mathbf{T}}\mathbf{T} = \kappa\mathbf{N},$$
$$\nabla_{\mathbf{T}}\mathbf{N} = -\kappa\mathbf{T} + \tau\mathbf{B},$$
$$\nabla_{\mathbf{T}}\mathbf{B} = -\tau\mathbf{N},$$

where κ is the curvature of γ and τ is its torsion,

$$g(\mathbf{T}, \mathbf{T}) = 1, g(\mathbf{N}, \mathbf{N}) = 1, g(\mathbf{B}, \mathbf{B}) = 1,$$
$$g(\mathbf{T}, \mathbf{N}) = g(\mathbf{T}, \mathbf{B}) = g(\mathbf{N}, \mathbf{B}) = 0.$$

Now we give a new frame different from Frenet frame. Let $\alpha : I \to \mathbb{S}^2_{Heis^3}$ be unit speed spherical curve. We denote σ as the arc-length parameter of α. Let us denote $\mathbf{t}(\sigma) = \alpha'(\sigma)$, and we call $\mathbf{t}(\sigma)$ a unit tangent vector of α. We now set a vector $\mathbf{s}(\sigma) = \alpha(\sigma) \times \mathbf{t}(\sigma)$ along α. This frame is called the Sabban frame of α on the Heisenberg group Heis^3 Then we have the following spherical Frenet-Serret formulae of α:

$$\nabla_{\mathbf{t}}\alpha = \mathbf{t},$$
$$\nabla_{\mathbf{t}}\mathbf{t} = -\alpha + \kappa_g \mathbf{s}, \tag{2.3}$$
$$\nabla_{\mathbf{t}}\mathbf{s} = -\kappa_g \mathbf{t},$$

where κ_g is the geodesic curvature of the curve α on the $\mathbb{S}^2_{Heis^3}$ and

$$g(\mathbf{t}, \mathbf{t}) = 1, g(\alpha, \alpha) = 1, g(\mathbf{s}, \mathbf{s}) = 1,$$
$$g(\mathbf{t}, \alpha) = g(\mathbf{t}, \mathbf{s}) = g(\alpha, \mathbf{s}) = 0.$$

With respect to the orthonormal basis $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, we can write

$$\alpha = \alpha_1\mathbf{e}_1 + \alpha_2\mathbf{e}_2 + \alpha_3\mathbf{e}_3,$$
$$\mathbf{t} = t_1\mathbf{e}_1 + t_2\mathbf{e}_2 + t_3\mathbf{e}_3,$$
$$\mathbf{s} = s_1\mathbf{e}_1 + s_2\mathbf{e}_2 + s_3\mathbf{e}_3.$$

To separate a biharmonic curve according to Sabban frame from that of Frenet- Serret frame, in the rest of the paper, we shall use notation for the curve defined above as biharmonic $\mathbb{S}$-curve, [9].

Theorem 2.1. Let $\alpha : I \to \mathbb{S}^2_{Heis^3}$ be a unit speed non-geodesic biharmonic $\mathbb{S}$-curve. Then, the parametric equations of α are

$$x(\sigma) = -\frac{\sin^2 \mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma + \mathsf{M}_1] + \mathsf{M}_2,$$

$$y(\sigma)=\frac{\sin^2 \mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3, \qquad (2.4)$$
$$z(\sigma)=\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}$$
$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4,$$

where $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3,\mathsf{M}_4$ are constants of integration and

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E})\text{and}\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E}.$$

3. PARALLEL SURFACES TO $\mathsf{S}-\alpha$ SURFACES OF BIHARMONIC S-CURVES ACCORDING TO SABBAN FRAME IN THE HEISENBERG GROUP HEIS^3

The purpose of this section is to study parallel surfaces to $\mathsf{S}-\alpha$ surface of biharmonic S-curve in the Heisenberg group Heis^3.

The $\mathsf{S}-\alpha$ surface of γ is a ruled surface

$$\mathsf{O}^{\mathsf{S}}(s,u)=\alpha(\sigma)+u\alpha(\sigma). \qquad (3.1)$$

Theorem 3.1. Let O^{S} be a $\mathsf{S}-\alpha$ surface of a unit speed non-geodesic biharmonic S-curve in the Heisenberg group Heis^3. Then, the parametric equations of $\mathsf{S}-\alpha$ surface of α are

$$\mathbf{x}_{\mathsf{O}^{\mathsf{S}}}(\sigma,u)=(1+u)[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2],$$
$$\mathbf{y}_{\mathsf{O}^{\mathsf{S}}}(\sigma,u)=(1+u)[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3], \qquad (3.2)$$
$$\mathbf{z}_{\mathsf{O}^{\mathsf{S}}}(\sigma,u)=(1+u)[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}$$
$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4],$$

where $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3,\mathsf{M}_4$ are constants of integration and

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E})\text{and}\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E}.$$

A parallel surface to $\mathsf{S}-\alpha$ surface is a parametrized surface

$$\mathbf{A}_{\mathsf{O}^{\mathsf{S}}}(s,u)=\mathsf{O}^{\mathsf{S}}(s,u)+\hat{\mathsf{o}}\mathbf{N}(s,u),$$

where $\hat{\mathsf{o}}$ is a constant.

Firstly, we need following lemma.

Lemma 3.2. Let O^{S} be a $\mathsf{S}-\alpha$ surface of a unit speed non-geodesic biharmonic S-curve in the Heisenberg group Heis^3. Then, the normal vector of $\mathsf{S}-\alpha$ surface of α is

$$\mathbf{N}_{\mathsf{O}^{\mathsf{S}}}=\frac{(1+u)}{\kappa_g}[\sin\mathsf{E}\cos[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]\mathbf{e}_1$$

$$+\frac{(1+u)}{\kappa_g}[-\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})+\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]\mathbf{e}_2$$
$$+\frac{(1+u)}{\kappa_g}[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E} \qquad (3.3)$$
$$-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$
$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4]\mathbf{e}_3,$$

where $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3,\mathsf{M}_4$ are constants of integration and

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E})\text{and}\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E}.$$

Now, we can prove the following interesting main result.

Theorem 3.3. Let O^{S} be a $\mathsf{S}-\alpha$ surface of a unit speed non-geodesic biharmonic S-curve in the Heisenberg group Heis^3. Then, equation of parallel surface to $\mathsf{S}-\alpha$ surface of α is

$$\mathbf{A}_{\mathsf{O}^{\mathsf{S}}}=(1+u)[[-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$
$$+\frac{\hat{\mathsf{o}}}{\kappa_g}[\sin\mathsf{E}\cos[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]]\mathbf{e}_1$$
$$+(1+u)[[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3] \qquad (3.4)$$
$$+\frac{\hat{\mathsf{o}}}{\kappa_g}[-\sin\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1](\mathsf{M}+\cos\mathsf{E})+\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3]]\mathbf{e}_2$$
$$+(1+u)[[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}$$
$$-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$
$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4]$$
$$+\frac{\hat{\mathsf{o}}}{\kappa_g}[\cos\mathsf{E}\sigma-\frac{\mathsf{V}\sigma+\mathsf{M}_1}{2\mathsf{V}^2}\sin^4\mathsf{E}-\frac{\sin 2[\mathsf{M}\sigma+\mathsf{M}_1]}{4\mathsf{V}^2}\sin^4\mathsf{E}$$
$$-[\frac{\sin^2\mathsf{E}}{\mathsf{V}}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_3][-\frac{\sin^2\mathsf{E}}{\mathsf{V}}\cos[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_2]$$
$$+\frac{\mathsf{M}_2}{\mathsf{V}}\sin^3\mathsf{E}\sin[\mathsf{M}\sigma+\mathsf{M}_1]+\mathsf{M}_4]]\mathbf{e}_3,$$

where $\mathsf{M}_1,\mathsf{M}_2,\mathsf{M}_3,\mathsf{M}_4$ are constants of integration and

$$\mathsf{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathsf{E}}-\cos\mathsf{E})\text{ and }\mathsf{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin 2\mathsf{E}.$$

Proof: By using (3.2) and (3.3) we obtain (3.4). Hence the proof is completed.

Theorem 3.4. Let O^{S} be a $\mathrm{S}-\alpha$ surface of a unit speed non-geodesic biharmonic S-curve in the Heisenberg group Heis^3. Then, equation of parallel surface to $\mathrm{S}-\alpha$ surface of α are

$$x=(1+u)[[-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]$$

$$+\frac{\hat{\mathrm{o}}}{\kappa_g}[\sin\mathrm{E}\cos[\mathrm{M}\sigma+\mathrm{M}_1](\mathrm{M}+\cos\mathrm{E})-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]],$$

$$y=(1+u)[[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]$$

$$+\frac{\hat{\mathrm{o}}}{\kappa_g}[-\sin\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1](\mathrm{M}+\cos\mathrm{E})+\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]],$$

$$z=(1+u)[[-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]$$

$$+\frac{\hat{\mathrm{o}}}{\kappa_g}[\sin\mathrm{E}\cos[\mathrm{M}\sigma+\mathrm{M}_1](\mathrm{M}+\cos\mathrm{E})-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]]$$

$$(1+u)[[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]$$

$$+\frac{\hat{\mathrm{o}}}{\kappa_g}[-\sin\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1](\mathrm{M}+\cos\mathrm{E})+\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3]]$$

$$+(1+u)[[\cos\mathrm{E}\sigma-\frac{\mathrm{V}\sigma+\mathrm{M}_1}{2\mathrm{V}^2}\sin^4\mathrm{E}-\frac{\sin2[\mathrm{M}\sigma+\mathrm{M}_1]}{4\mathrm{V}^2}\sin^4\mathrm{E}$$

$$-[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3][-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]$$

$$+\frac{\mathrm{M}_2}{\mathrm{V}}\sin^3\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_4]$$

$$+\frac{\hat{\mathrm{o}}}{\kappa_g}[\cos\mathrm{E}\sigma-\frac{\mathrm{V}\sigma+\mathrm{M}_1}{2\mathrm{V}^2}\sin^4\mathrm{E}-\frac{\sin2[\mathrm{M}\sigma+\mathrm{M}_1]}{4\mathrm{V}^2}\sin^4\mathrm{E}$$

$$-[\frac{\sin^2\mathrm{E}}{\mathrm{V}}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_3][-\frac{\sin^2\mathrm{E}}{\mathrm{V}}\cos[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_2]$$

$$+\frac{\mathrm{M}_2}{\mathrm{V}}\sin^3\mathrm{E}\sin[\mathrm{M}\sigma+\mathrm{M}_1]+\mathrm{M}_4]],$$

where $\mathrm{M}_1,\mathrm{M}_2,\mathrm{M}_3,\mathrm{M}_4$ are constants of integration and

$$\mathrm{M}=(\frac{\sqrt{1+\kappa_g^2}}{\sin\mathrm{E}}-\cos\mathrm{E})\text{and}\mathrm{V}=\sqrt{1+\kappa_g^2}-\frac{1}{2}\sin2\mathrm{E}.$$

Proof: From basis vectors we easily get above system.

Thus, the following figures are valid.

If we use Mathematica for different constant, yields

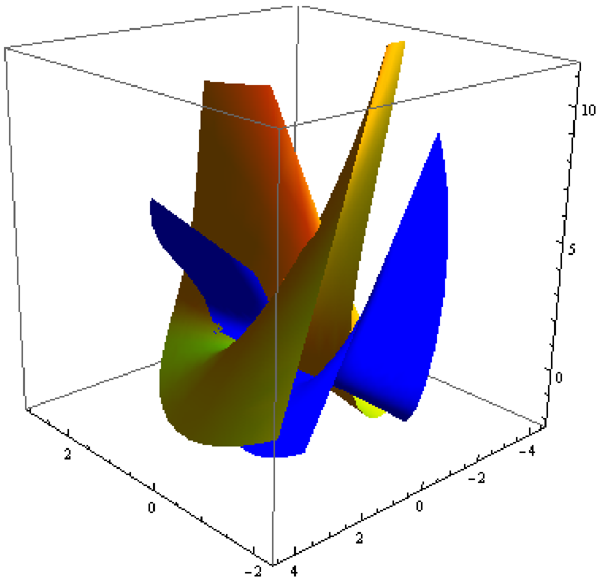

Fig. 1.

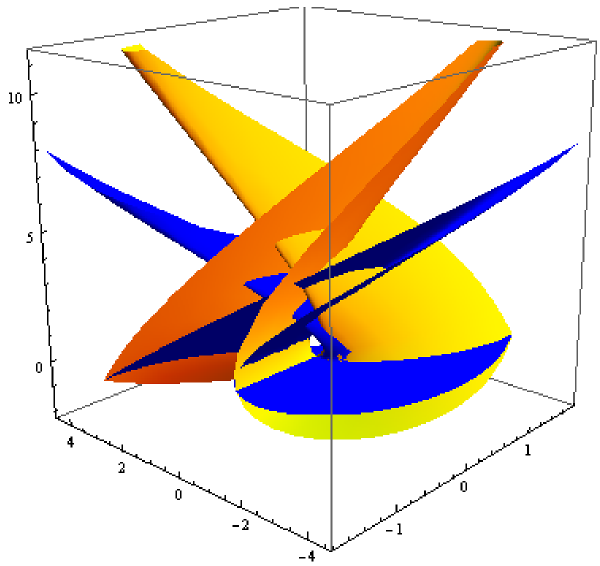

Fig. 2.

Figs. 1, 2. The equations of $S-\alpha$ surface of a unit speed non-geodesic biharmonic $S-$ curve and its parallel surface are illustrated colour Blue, Yellow’ respectively.

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