ORIGINAL PAPER

APPLICATION OF TOTAL LEAST SQUARES TO A LINEAR SURVEYING NETWORK

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Abstract. *Despite the classical least squares being the de-facto technique for adjusting Surveying networks, this research explores the application of total least squares to solving a linear surveying network problem. The linear surveying network used for the experiment is a three-loop levelling network. The augmented matrix of the design matrix and observation vector is first computed. Thereafter the singular value decomposition of the augmented matrix of the design matrix and the vector of unknown parameters are obtained. The residuals from the total least squares when compared with those from the classical least squares, are relatively better.*

Keywords: *total least squares, least squares, singular value decomposition, linear surveying network*

1. INTRODUCTION

Total Least Squares (TLS) is a relatively new algorithm; while the Least Squares (LS) is the classical technique applied in the field of Surveying for the adjustment of Surveying networks. The objective of this work therefore is to illustrate the application of TLS to solving a linear Surveying problem. The result from the TLS will be compared with the result from the classical LS approach.

2. TOTAL LEAST SQUARES

Fundamentally an LS estimate for variable $\breve{x}$ can be expressed as,

$$\breve{x} = (A^T A)^{-1} A^T b \tag{1}$$

here $\breve{x}$ represents vector of unknown parameters; A is the design matrix; while b denotes the vector of observations or the target vector . In the case of TLS, it assumes that all the elements of the data are erroneous; this situation can be stated mathematically as,

$$b + \Delta b = (A + \Delta A)x\,,\ rank(A) = m < n \tag{2}$$

where, Δb is error vector of observations and ΔA is error matrix of data matrix A. Both errors are assumed independently and identically distributed with zero mean and with same variance [1].

The estimation procedure is an optimisation problem given by,

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$$\text{minimise } \left\|[A;b]-[\hat{A};\hat{b}]\right\|_F \quad . \quad [\hat{A};\hat{b}] \in R^{n(m+1)} \tag{3}$$

subject to: $b + \Delta b = (A + \Delta A)\breve{x}$; where m is the number of unknowns; and n is the number of observations. Once a minimising $[\hat{A};\hat{b}]$ is found, any $\breve{x}$ that satisfies $\hat{A}\breve{x} = \hat{b}$ is a called TLS solution and $[\Delta\hat{A};\Delta\hat{b}] = [A;b] - [\hat{A} - \hat{b}]$ is the corresponding TLS correction [2-6]. From equation 3, $\| \ \|_F$ denotes the Frobenius norm. The basic TLS problem given in equation 3 can be solved using Singular Value Decomposition (SVD) [2, 4, 6]. For the solution of $A\breve{x} \approx b$, we write the functional relation as follows:

$$[A;b] \ [\breve{x}^T;-1]^T \approx 0 \tag{4}$$

Therefore SVD of the augmented matrix $[A;b]$ is computed as follows:

$$[A;b] = U\Sigma V^T \tag{5}$$

Where, $U = [U_1;U_2]$, $U_1 = [u_1,...,u_m]$, $U_2 = [u_{m+1},...,u_n]$ and $u_i \in R^n$,

$U^T U = I_n$. $V = [v_1,...,v_m,v_{m+1}], v_i \in R^{m+1}$, V^T

$V = I_{m+1}.\sum = diag(\sigma_1,...,\sigma_m,\sigma_{m+1}) \in R^{n\times(m+1)}$.

In eq. 5, the rank of the matrix $[A;b]$ is $m+1$ which must be reduced to m. For the purpose, Eckart-Young Minsky theorem is used [7]. After the rank reduction, the solution of the basic TLS is obtained by,

$$[\breve{x}^T;-1]^T = \frac{-1}{V_{m+1,m+1}} v_{m+1} \tag{6}$$

If $V_{m+1,m+1} \neq 0$, then $\hat{b} = \hat{A}\breve{x} = -1/(V_{m+1,m+1})\,\hat{A}[V_{1,m+1},...,V_{m,m+1}]^T$ which belongs to the column space of $\hat{A}$, and hence $\breve{x}$ solves the basic TLS problem [4].

3. APPLICATION

In order to adjust the orthometric heights of points B, C, and D (H_B, H_C, and H_D), given the adjusted height of point A, H_A^a=100m, the following data (of a linear Surveying network) expressed in the table and diagram below were observed [8]:

Table 1. Levelling data.

From	To	Change in height (m): Δh_i, $i = 1,2,...,6$; Rise(+), Fall (-)
A	C	+6.16(Δh_1)
A	D	+12.57(Δh_2)
A	B	+1.09(Δh_3)
C	D	+6.43(Δh_4)
B	D	+11.58(Δh_5)
B	C	+5.07(Δh_6)

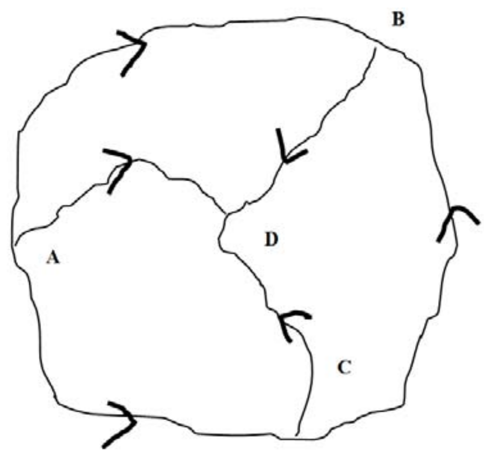

Fig. 1. Pictorial illustration of the levelling data given in Table 1.

Given below are the six observation equations derived from Table 1 and Fig. 1:

$$\Delta h_1^a = H_C^a - H_A^a \quad (7)$$

$$\Delta h_2^a = H_D^a - H_A^a \quad (8)$$

$$\Delta h_3^a = H_B^a - H_A^a \quad (9)$$

$$\Delta h_4^a = H_D^a - H_C^a \quad (10)$$

$$\Delta h_5^a = H_D^a - H_B^a \quad (11)$$

$$\Delta h_6^a = H_C^a - H_B^a \quad (12)$$

Where H_B^a, H_C^a, and H_D^a are the adjusted heights of points B, and C, and D. Eqs. 7-12 can be re-rewritten as equations 13-18 after substituting the known adjusted height of point A,

$$106.16 = H_C^a \quad (13)$$

$$112.57 = H_D^a \quad (14)$$

$$101.09 = H_B^a \quad (15)$$

$$6.43 = H_D^a - H_C^a \quad (16)$$

$$11.58 = H_D^a - H_B^a \quad (17)$$

$$5.07 = H_C^a - H_B^a \quad (18)$$

By differentiating the right-hand side of equations 13-18 with respect to H_B^a, H_C^a, and H_D^a the expression for the design matrix A was derived as,

$$A = \frac{\partial f(\breve{x}^0)}{\partial \breve{x}^a} = \begin{vmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & -1 & 1 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{vmatrix}$$

. Where, $\breve{x}^0$ and $\breve{x}^a$ represent the approximate and adjusted values of the unknown parameters H_B, H_C, and H_D. The left-hand side of equations 13-18 is the observation vector b,

$$b = \begin{vmatrix} 106.16 \\ 112.57 \\ 101.09 \\ 6.43 \\ 11.58 \\ 5.07 \end{vmatrix}$$

. A is augmented with b to obtain C, $C = \begin{vmatrix} 0 & 1 & 0 & 106.1600 \\ 0 & 0 & 1 & 112.5700 \\ 1 & 0 & 0 & 101.0900 \\ 0 & -1 & 1 & 6.4300 \end{vmatrix}$. The result of the find the SVD of C is,

$$V = \begin{vmatrix} -0.002460 & 0.455159 & 0.702737 & -0.546790 \\ -0.003050 & -0.814630 & 0.080733 & -0.574330 \\ -0.003800 & 0.359466 & -0.706850 & -0.609200 \\ -0.999990 & -6.96e-16 & 0.000713 & 0.00541 \end{vmatrix}$$

The computed solution for the unknown parameters H_B^a, H_C^a, and H_D^a is,

$$\breve{x} = \begin{vmatrix} H_B^a \\ H_C^a \\ H_D^a \end{vmatrix} = \begin{vmatrix} 101.0650 \\ 106.1550 \\ 112.6000 \end{vmatrix}$$

. The residual r is computed as $r = A\breve{x} - b$ for TLS and LS are presented in Table 2.

Table 2. Computed residuals for TLS and LS.

TLS	LS
-0.004986907571578	-0.004999999999995
0.030013290517402	0.030000000000001
-0.024987064013899	-0.025000000000006
0.015000198088977	0.014999999999993
-0.044999645468708	-0.045000000000003
0.020000156442315	0.020000000000003

In order to decipher the more accurate model between the TLS and LS, the formula, $sum(abs(A * \breve{x} - b))$ was used to obtain a single accuracy value for the TLS and LS, computed as 0.139987262102879 and 0.140000000000002 for TLS and LS respectively. TLS yielded a lower residual value and therefore marginally better than the LS.

4. CONCLUSION

The classical LS technique offers the best least unbiased estimate; and is renowned for eliminating random errors from Surveying measurements; however this work has shown that TLS can yield better results than the classical LS.

REFERENCES

[1] Dogan, M.O., Altan, M.O., Total least squares registration of images for change detection. *ISPRS Archive* Vol. XXXVIII, Part 4-8-2-W9, Core Spatial Databases – Updating, Maintenance and Services – from Theory to Practice, Haifa, Israel, 2010.

[2] Golub, G.H., Loan, C.F.V., *SIAM Journal of Numerical Analysis,* **17,** 883, 1980.

[3] Akyilmaz, O., *Survey Review*, **39**, 68, 2007.

[4] Huffel, S.V., Vandewalle, J., *The Total Least Squares Problem: Computational Aspects and Analysis*, 1st Ed., SIAM, Philadelphia, 1991.

[5] Golub, G.H., Reinsch, C., *Numerische Mathematik*, **14**, 403, 1970.

[6] Golub, G.H., *SIAM Review*, **15**, 318, 1973.

[7] Eckart, G., Young, G., *Psychometrica*, **1**, 211, 1963.

[8] Ayeni, O.O., *Statistical adjustment and analysis of data, A Manual*, in Department of Surveying and Geoinformatics, Faculty of Engineering, University of Lagos, Lagos, Nigeria, 2001.