GENERAL EXISTENCE FOR MINIMUM PROBLEMS IN BANACH SPACES

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Abstract. *In this note we give some extensions of a recent result concerning the minimum problems in Banach spaces. These extensions are based on the fact that the topological degree of a coercive map is one and this still remains true for a pseudomonotone potential map between a reflexive Banach space and its dual space. Some considerations concerning the minimum problem and the solution property of degree required in applications are also appended.*

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1. ABSTRACT MINIMUM PROBLEMS

Let X be a real reflexive Banach space, $K \subseteq X$ be a non-empty closed convex set and $\phi : X \mapsto]-\infty, +\infty]$ be a proper l.s.c. function, that is Gâteaux differentiable on K. Denote by $F : K \mapsto X^*$ its Gâteaux gradient $F := \nabla \phi$. The minimum problem:

$$\inf \left\{ \phi(x); \, x \in K \right\} \tag{1}$$

has a solution by Weierstrass theorem when K is bounded and weakly closed (see [6, p.24]). When ϕ is convex, then the above problem is equivalent with the variational inequation:

$$\left\langle F(x), y - x \right\rangle \geq 0, \, (\forall) y \in K \tag{2}$$

and we can extend the existence of solutions by using topological arguments such as fixed-point theorems or the solution property of the topological degree.

The key role in definition of the topological degree for mappings of monotone type is played by the so called *mappings of type* (S_+) and the *pseudomonotone* ones. The operator T is of type (S_+), denoted $T \in (S_+)$, if each sequence $\{u_n\} \subset X$ with $u_n \rightarrow u_0$ for which

$$\lim \, \sup \left\langle Tu_n, \, u_n - u_0 \right\rangle \leq 0 \tag{3}$$

is in fact strongly convergent in X.

The operator T is *pseudomonotone*, denoted $T \in (PM)$, if each sequence $\{u_n\}$ in X with $u_n \rightarrow u_0$ for which (3) holds, it follows that $\left\langle Tu_n, u_n - u_0 \right\rangle \rightarrow 0$ and $Tu_n \rightarrow Tu_0$. Moreover, T is *quasimonotone*, denoted $T \in (QM)$, if for each sequence $\{u_n\}$ in X with $u_n \rightarrow u_0$, it follows that

$$\lim \, \sup \left\langle Tu_n - Tu_0, \, u_n - u_0 \right\rangle \geq 0$$

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We assume that all mappings considered are *bounded*, i.e., they carry bounded sets into bounded sets, and *demicontinuous*, i.e., $Tu_n \rightharpoonup Tu_0$ for any $u_n \to u_0$. For a given open bounded set $\Omega \subset X$, let us denote:

$$F_\Omega(\cdot) := \left\{ F : \overline{\Omega} \to X^* / F \in (\cdot), \ bounded \ and \ demicontinuos \right\}.$$

Here $(\cdot)$ means that the operator belongs to one of above mentioned monotone type classes. Hence for any $p \in X^*$ with $p \notin F(\mathcal{J}\Omega)$ we can define (S_+) – degree as follows:

$$d_{S_+}(F, \Omega, p) = \lim_{\varepsilon \to 0} d_{LS}(F_\varepsilon, \Omega, p) \qquad (4)$$

Here the family of mappings $\left\{ F_\varepsilon : \overline{\Omega} \mapsto X; \ \varepsilon > 0 \right\}$ is constructed with the original map *F*:

$$F_\varepsilon(u) := u + \frac{1}{\varepsilon} \Psi \Phi F(u), u \in \overline{\Omega} \qquad (5)$$

with *H* a real Hilbert space and $\Phi : X^* \mapsto H$, $\Psi : H \mapsto X$ linear compact operators and adjoint each other as stated by Browder-Ton lemma. Moreover, F_ε are Leray-Schauder maps and $d_{LS}(F_\varepsilon, \Omega, p)$ is well defined for all $p \notin F_\varepsilon(\mathcal{J}\Omega)$ (see [11, p.14] for details).

Of course, the (S_+) – degree has all the properties of LS – degree, except the normalization property, when we take instead of *I* the duality map $J : X \to 2^{X^*}$, defined by $\|J(x)\| = \|x\|$ and $\langle J(x,x) \rangle = \|x\|^2$. For *X* a locally uniformly convex Banach space, *J* is an operator (an univalent mapping), $J : X \to X^*$, and belongs to class (S_+) as we can easily see ([6, p.145]).

Since *any demicontinuos operator $T : X \to X^*$ is quasimonotone if and only if for each $\mu > 0$ the mapping $T + \mu J$ is of type (S_+)* by Calvert-Webb lemma (see [11, p.16]), define:

$$d_{QM}(F, \Omega, P) = \lim_{\mu \to 0} d_{S_+}(F + \mu J, \Omega, p) \qquad (6)$$

with $p \notin \overline{F(\mathcal{J}\Omega)}$. This degree can not be a classical one, because in this case the image *F(A)* of a closed set $A \subseteq \overline{\Omega}$ is no more closed. This fact modifies all the properties; for instance, *the solution property* (see [11] for details) will be replaced in this case by the following implication

$$d_{QM}(F, \Omega, P) \neq 0 \Rightarrow p \in \overline{F(\Omega)} \qquad (7)$$

Since $(S_+) \subset (PM) \subset (QM)$, the (QM) – degree is well defined for all $F \in F_\Omega(PM)$. The interest of pseudomonotone mappings in applications is due to the following closeness property:

Proposition 1. For each $F \in F_\Omega(PM)$, the set *F(A)* is closed whenever $A \subseteq \overline{\Omega}$ is weakly closed.

Proof: Indeed, if $\{w_n\} \subset F(A)$ with $w_n \to w$, then $w_n = F(u_n)$ for some $\{u_n\} \subset A$. Since Ω is bounded, $u_n \rightharpoonup u$ for some $u \in A$ at least on a subsequence. Thus, $\lim \sup \langle F(u_n), \ u_n - u \rangle = 0$, implying $F(u_n) \to F(u)$ and hence $w = F(u) \in F(A)$. (q.e.d)

In particular, if Ω in convex, then $\overline{\Omega}$ is weakly closed, implying that $F(\overline{\Omega})$ is closed. Consequently, for $F \in F_\Omega(PM)$ and Ω convex, we can conclude:

$$d_{QM}(F,\Omega,p) \neq 0 \Rightarrow p \in F(\overline{\Omega}) \tag{8}$$

It is well known that (see [12]):

(i) ϕ *is convex on K* $\Leftrightarrow$ *F is monotone on K;*

(ii) $F \in (PM) \Rightarrow \phi$ *is l.s.c. on K* .

Moreover, by homotopy arguments (see [6, p.232]), we can prove:

(iii) $F \in (PM)$, *demicontinuous and coercive* $\Rightarrow (\exists)$ *at least one solution of the equation F(x) = 0.*

This justifies the application of the solution property for pseudomonotone mappings on convex sets, and assures to our extension from the next section, the maximum topological meaning.

2. PSEUDOMONOTONE POTENTIAL OPERATORS

Let U be an open set in a real reflexive Banach space X and $\phi \in C^1(u,R)$ a given function with $F = \nabla\phi : X \to X*$, the Gâteaux gradient, a pseudomonotone operator. Observe that we can work with Gâteaux gradients instead of Fréchet ones (see e.g. [12]). Taking into account the results from preceeding section and the solution property of the topological degree, we are interested in those conditions that assure the nonvanishing degree.

Firstly we have the following result, proved by Amann [1] for compact operators:

Theorem 1. Suppose that for some $\beta \in \mathbb{R}$, the set $V = \phi^{-1}(-\infty,\beta)$ is bounded and $V \subset U$. Moreover suppose that there are numbers $\alpha < \beta$ and $r > 0$ and a point $x_0 \in U$ such that $\phi^{-1}(-\infty,\alpha] \subset B(x_0,r) \subset V$ and $F(x) \neq 0$, $\forall x \in \phi^{-1}[\alpha,\beta]$. Then $d_{PM}(F,V,0) = 1$.

Proof: Since $V = \phi^{-1}(-\infty,\beta)$ is bounded and $\overline{V} \subset U$ it follows that $v = \Psi^{-1}(V)$ is relatively compact in H and thus there exist $u_0 \in H$ and $r > 0$ such that $v \subset B(u_0,r)$. Similarly, $\Psi(B(u_0,r))$ is relatively compact in X and $\overline{V} \subset U$, which implies that $\Psi(B(u_0,r)) \subset B(u_0,r) \subset U$ for $x_0 = \psi(u_0)$ and some $R > 0$.

Now, for any $\alpha < \beta$ the set $V_\alpha := \phi^{-1}(-\infty,\alpha] \subset V$ and thus it is bounded and closed. Then $\Psi^{-1}(V_\alpha)$ is compact and $\Psi^{-1}(V_\alpha) \subset B(u_0,r)$. Hence $V_\alpha \subset V \subset B(x_0,R) \subset U$.

Suppose that $F(x) \neq 0$, $\forall x \in f^{-1}[\alpha,\beta]$. Then $\mathcal{F} := \Phi F \Psi : H \to H$ is compact in H and

$$|\mathcal{F}(u)| = \sup(\mathcal{F}(u),v) = \sup_{|v|=1}(\Phi F\Psi(u),v) = \sup_{|v|=1}\langle F\Psi(u),\Psi(v)\rangle =$$
$$= \|F(\Psi u)\| \cdot \|\Psi\| = \|F(x)\| > 0, \forall x = \Psi u \in \phi^{-1}[\alpha,\beta]. \tag{9}$$

As $F \in (PM) \Rightarrow (F + \mu J) \in (S_+)$ and thus $\mathcal{F}_{\varepsilon\mu} := I + \dfrac{1}{\varepsilon}\Phi(F + \mu J)\Psi \in LS(H)$ and $d_{LS}(\mathcal{F}_{\varepsilon\mu},v,0) = 1$. Hence

$$d_{S_+}(F + \mu J,V,0) = \lim_{\varepsilon \to 0} d_{LS}(\mathcal{F}_{\varepsilon\mu},v,0) = 1$$

and

$$d_{PM}(F,V,0) = \lim_{\mu \to 0}(F + \mu J, V, 0) = 1 \text{ by definition. (q.e.d)}$$

Remark that the result is generally true for mappings of class *(QM)*, but we are restricted to the class *(PM)* because of the solution property as stated in the previous section.

We shall prove that all direct consequences still remain true also for operators of class *(PM)*.

Corollary 1. Let ϕ be a real C^l – function on X such that its Gâteaux gradient $F = \nabla\phi : X \to X^*$ is a bounded map of class *(PM)*. Suppose that $\phi(x) \to \infty$ as $\|x\| \to \infty$ and that $F(x) \neq 0$ for all $\|x\| > r_0$ and some $r_0 > 0$. Then there exists $r_1 \geq r_0$ such that $d_{PM}(F, B(0,r), 0) = 1, \ \forall r \geq r_1$.

Proof: Observe that $\phi(x) = \int_0^1 \langle F(sx), x \rangle ds$. Take $x = \Psi u$ and write

$$\phi(\Psi u) = (\phi\Psi)(u) = \int_0^1 \langle F(s\Psi u), \Psi u \rangle ds = $$
$$= \int_0^1 \langle \Phi F \Psi(su), u \rangle ds = \int_0^1 \langle \mathcal{F}(su), u \rangle ds \tag{10}$$

with $\mathcal{F}$ a compact operator on H. Here $\phi\Psi$ maps bounded sets from H into bounded sets in $\mathbb{R}$ and $\Psi(B(0,r))$ is a compact set in X because $\Psi : H \to X$ is a compact linear map. Hence there exists an $r_0 > 0$ such that $\Psi(B(0,\delta)) \subseteq B(0,r_0) \subset X$.

Thus, let $\alpha = \sup\phi(B(0,r_0))$ and $r_1 = \sup\{\|x\|; x \in \phi^{-1}((-\infty, \alpha])\}$. For given $r \geq r_1$, fix $\beta > \sup\phi(B(0,r))$ and apply the theorem with $x_0 = 0$ and the excision property of degree. (q.e.d)

Proposition 2. Suppose that $\phi : X \to \mathbb{R}$ is a C^l function on an open convex set $U \subseteq X$ with its Gâteaux gradient $F = \nabla\phi : X \to X^*$ of class *(PM)* on U. Let $x_0 \in U$ be an isolated critical point of ϕ on U. If fi has a local minimum at x_0, then $i(F, x_0) = 1$.

Proof: Since $F \in (PM)$ it follows that fi is weakly sequentially lower semicontinuous on U (see [6], p.24). We can suppose without loss of generality that $x_0 = 0$, $\phi(0) = 0$ and there exists an $r_0 > 0$ such that $U = B(0,r_0)$ and 0 is the unique critical point of ϕ in U.

We claim that
$$\inf\phi(\overline{B}(0,r_2) \setminus B(0,r_1)) > 0 \tag{11}$$
whenever $0 < r_1 \leq r_2 < r_0$.

Indeed, otherwise it would exist a sequence $\{x_n\} \subset \overline{B}(0,r_2) \setminus \overline{B}(0,r_1)$ such that $\phi(x_n) \to 0$. As X is a reflexive Banach space and $\{x_n\}$ is bounded we can assume that $x_n \to x \in \overline{B}(0,r_2)$ at least on a subsequence. Since ϕ is weakly sequentially lower semicontinuous it results the implication
$$0 \leq \phi(x) \leq \liminf\phi(x_n) = 0 \Rightarrow x = 0 \tag{12}$$

On the other hand, for any $n \in \mathbb{N}$ by the Lagrange's formula there is an $s_n \in (0,1)$ such that

$$\phi(x_n) - \phi\left(\frac{x_n}{2}\right) = \left\langle F\left(\frac{x_n}{2} + s_n \frac{x_n}{2}\right), \frac{x_n}{2} \right\rangle$$

Since $\phi\left(\frac{x_n}{2}\right) \geq 0$ and $\lim \phi(x_n) = 0$, by (11) we get

$$0 \geq \limsup \left\langle F\left(\frac{x_n}{2} + s_n \frac{x_n}{2}\right), \frac{x_n}{2} \right\rangle =$$

$$= \limsup \left\langle F\left((1+s_n)\frac{x_n}{2}\right), (1+s_n)\frac{x_n}{2} - s_n \frac{x_n}{2} \right\rangle.$$

As $F \in (PM)$, it follows that $F + \mu J \in (S_+)$ and

$$\limsup \left\langle F\left((1+s_n)\frac{x_n}{2}\right) + \mu J\left((1+s_n)\frac{x_n}{2}\right), (1+s_n)\frac{x_n}{2} - s_n \frac{x_n}{2} \right\rangle =$$

$$= \limsup \left\langle F\left((1+s_n)\frac{x_n}{2}\right), (1+s_n)\frac{x_n}{2} - s_n \frac{x_n}{2} \right\rangle + \mu \limsup \left\langle J\left((1+s_n)\frac{x_n}{2}\right), (1+s_n)\frac{x_n}{2} - s_n \frac{x_n}{2} \right\rangle \leq$$

$$\leq \mu \limsup \left[\left\langle J\left((1+s_n)\frac{x_n}{2}\right), (1+s_n)\frac{x_n}{2} \right\rangle - \left\langle J\left((1+s_n)\frac{x_n}{2}\right), s_n \frac{x_n}{2} \right\rangle \right] \leq$$

$$\leq \mu \limsup \left[\left\| (1+s_n)\frac{x_n}{2} \right\|^2 - \left\| J\left((1+s_n)\frac{x_n}{2}\right) \right\| \cdot \left\| s_n \frac{x_n}{2} \right\| \right] \leq$$

$$\leq -\mu \limsup \left\| (1+s_n)\frac{x_n}{2} \right\| \cdot \left\| s_n \frac{x_n}{2} \right\| \leq 0.$$

Hence $(1+s_n)\frac{x_n}{2} \to 0$ and also $x_n \to 0$, in contradiction with $\|x_n\| \geq r_1$. Now, fix r_1 and r_2 with $0 < r_1 < r_2 < r_0$, take $\beta = \inf \phi\left(\overline{B}(0,r_2) \setminus B(0,r_1)\right)$ and choose $r > 0$ such that $\overline{V} = \overline{B}(0,r) \subset \phi^{-1}(-\infty, \beta)$. The result follows by applying theorem 2 with $U = B(0,r_2)$ and $\alpha = \frac{1}{2} \inf \phi\left(\overline{B}(0,r_2) \setminus B(0,r)\right)$ since $i_{PM}(F,0) = d_{PM}(F,V,0)$. (q.e.d)

Theorem 2. Under the hypotheses of theorem 1, suppose that $x_1 \in V$ is a critical point of ϕ, which is not a global minimum of ϕ in V, such that either F is Fréchet differentiable at x_1 and $\lambda = 1$ is not an eigenvalue of $F'(x_1) \in \mathcal{L}(X, X^*)$ or x_1 is a local minimum. Then ϕ has at least three critical points in V. The assertion remains true in the case $U = X$ and $\phi(x) \to \infty$ as $\|x\| \to \infty$.

Proof: As $F \in (PM) \Rightarrow (F + \mu J) \in (S_+)$ and thus $\mathcal{F}_\mu = \Phi(F + \mu J)\Psi$ is compact and also of (S_+) type on H. By chain rule and the Krasnoselsskii's theorem

$$\mathcal{F}_\mu'(u_1) = \Phi'(F + \mu J)'\Psi'(u_1) = \Phi(F + \mu J)'(x_1)$$

(with $x_1 = \Psi'(u_1)$) is a compact linear operator on H and, thus, it is a continuous automorphism of H because $\lambda = 1$ is not an eigenvalue of $\mathcal{F}'_{\mu}(u_1)$. Hence x_1 is an isolated zero of F and $i(\Phi F, x_1) = \pm 1$ by the Leray-Schauder index formula.

Since $F \in (PM)$ it follows that ϕ is weakly sequentially lower semicontinous on V and it attains its global minimum at some $x_2 \in V$ by the Weienstrass theorem (see [6, p.24]) and $x_1 \neq x_2$. If x_1 and x_2 are the only critical points in V then

$$1 = d_{LS}(\Phi F, V, 0) = i(\Phi F, x_1) + i(\Phi F, x_2) = 0 \text{ or } 2$$

by the additivity theorem of LS-degree (see [8, p.73]), which is a contradiction.

Thus, ϕ must have at least three critical points in V. The last statement follows by Corollary 1. (q.e.d)

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