ORIGINAL PAPER

ON A CLASS OF POSITIVE LINEAR OPERATORS

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Abstract. *A new class of positive linear operators have been introduced which contains a number of well known positive linear operators such as Gamma-Operators of Muller, Post-Widder and Modi.ed Post-Widder Operators as particular cases.Some basic approximation properties of this class of operators have been studied in this paper.*

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1. INTRODUCTION

A number of classes and sequences of positive linear operators (henceforth written as operator) both, of the summation and those defined by integrals have been introduced and studied by a number of authors during the past few decades. Some of well known operators of latter type are the Gamma-Operators of Muller [10], Post-Widder Operators [16], Modified Post-Widder Operators [9], Gauss-Weierstrass Integrals [13], Convolution type operators [14], Baskakov Operators [1], and the operators studied by De Vore [2], Leviaton [8], Kunwar [7], Sikkema and Rathore [15].

In this paper we will study a class of operators which contains a number of well known operators as special cases. This class of operators was introduced in Kunwar [7]. Now we will give a brief description of the notations and definitions followed by the definition of the operators.

Throughout the paper IR^+ denotes the interval $(0,\infty)$, $\langle a,b\rangle$ open interval containing $[a,b]\subseteq IR^+$, $\chi_{\delta,x}\left(\chi_{\delta,x}^c\right)$ the characteristic function of the interval $(x-\delta,x+\delta)$ $\left\{IR^+-(x-\delta,x+\delta)\right\}$. The spaces $M\left(IR^+\right)$, $M_b\left(IR^+\right)$, $Loc\left(IR^+\right)$, $L^1\left(IR^+\right)$ respectively denote the sets of complex valued measurable, bounded and measurable, locally integrable and Lebesgue integrable functions on IR^+.

Now we de.ne our operator L_n [7] and give some elementary properties of the same.

$$L_n(f;x)=D(m,n,\alpha)x^{mn+\alpha-1}\int_0^\infty u^{-mn-\alpha}e^{-n\left(\frac{x}{u}\right)^m}f(u)du$$

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where $D(m,n,\alpha)=\dfrac{|m|n^{n+\frac{\alpha-1}{m}}}{\Gamma\left(n+\frac{\alpha-1}{m}\right)}$, $m\in IR-\{0\},\ n>0,\ \alpha\in IR$.

Several well known operators are special cases of L_n:

Choosing $m=1$ and $\alpha=2$, the operator reduces to the Gamma-Operators of Muller [10] denoted and defined by

(i) $$G_n(f;x)=\frac{x^{n+1}}{n!}\int_0^{\infty}t^n e^{-tx} f\left(\frac{n}{t}\right)dt$$

Choosing $m=-1,\ \alpha=1$ and $m=-1,\ \alpha=0$ and by proper substitution the operators Ln reduces respectively to

(ii) the Post-Widder operators S_n^1 May [9] defined by,

$$S_n^1(f;t)=\frac{1}{(n-1)!}\int_0^{\infty}e^{-n\frac{u}{t}}u^{n-1}f(u)\,du$$

and

(iii) the operators $L_{k,t}$ (Widder [16]) defined by

$$L_{k,t}(f;x)=\frac{1}{k!}\left(\frac{k}{t}\right)^{k-1}\int_0^{\infty}e^{-\frac{ku}{t}}u^k f(u)\,du\,.$$

We will make use of a bounding function introduced by Rathore [13] for establishing the basic convergence theorem for our operators.

Definition 1. Let $\Omega(>1)$ be a continuous function defined on IR^+. We call Ω, a bounding function if for each compact $K\subseteq IR^+$, there exist positive numbers n_k and M_k such that

$$L_{n_k}(\Omega;x)<M_k,\ x\in K$$

For our operators the bounding function is

$$\Omega(u)=u^{-a}+e^{bu^m}+u^c\text{, where } a,b,c>0\,.$$

For this bounding function Ω, we define

$$D_\Omega=\left\{f\in Loc\left(IR^+\right)\text{ s.t. }\limsup_{u\to 0}\frac{f(u}{\Omega(u)}\text{ and }\limsup_{u\to\infty}\frac{f(u}{\Omega(u)}\text{ exist}\right\}.$$

2. BASIC APPROXIMATION

Lemma 1. If $0<\delta<a<b<\infty$ and $f\in D_\Omega$, then

$$\lim_{n\to\infty} n^k L_n\left(f\chi^c_{\delta,x};x\right)=0 \tag{2.1}$$

uniformly in $x\in[a,b]$ for any $k\in IR^+$.

Proof: Since $f\in D_\Omega$, $\exists$ positive constants *A*, *B* and *M* such that $A<\min\{1,a\}$ and $B>\max\{1,b\}$ and $|f(u)|<M\Omega(u)$ for all $u\in\left(0,\frac{b}{B}\right)\cup\left(\frac{a}{A},\infty\right)$. Let $J(A,B)=(0,A)\cup(B,\infty)$ then

$$\left|\int_{J(A,B)} u^{mn+\alpha-2}e^{-nu^m} f\left(\frac{x}{u}\right)\chi^c_{\delta,x}\left(\frac{x}{u}\right)du\right|\le M\int_{J(A,B)} u^{mn+\alpha-2}e^{-nu^m}\Omega\left(\frac{x}{u}\right)du \tag{2.2}$$

For $\Omega(u)$, there exists n_1, $M_1>0$ such that $L_{n_1}(\Omega;x)<M_1$, for all $x\in[a,b]$.

For any $\varepsilon>0$ we have

$$u^m e^{-u^m}<\frac{1}{e}-2\varepsilon,$$

for almost all $u\in J(A,B)$.

Hence if $n>n_0>n_1$, we have

$$\begin{aligned}\int_{J(A,B)} u^{mn+\alpha-2}e^{-nu^m}\Omega\left(\frac{x}{u}\right)du &\le\left(\frac{1}{e}-2\varepsilon\right)^{n-n_0}\frac{1}{D(m,n,\alpha)}L_{n_0}(\Omega;x)\le\\ &\le M_1\frac{1}{D(m,n,\alpha)}\left(\frac{1}{e}\right)^{n-n_0}\left(\frac{1}{e}-2\varepsilon\right)^{n-n_0}=\\ &=\left[\frac{M(n_0,n_1)}{M}\right]\left(\frac{1}{e}-2\varepsilon\right)^{n}\ (\text{say}).\end{aligned} \tag{2.3}$$

By choosing a positive δ_1 such that $\frac{b}{b+\delta}<1-\delta_1<1+\delta_1<\frac{b}{b-\delta}$ and using the property of the function $u^m e^{-u^m}$ for sufficiently small ε we have $u^m e^{-u^m}<\frac{1}{e}-2\varepsilon$ for almost all $u\in IR^+-(1-\delta_1,1+\delta_1)$.

Hence

$$\left|\int_A^B u^{mn+\alpha-2}e^{-nu^m} f\left(\frac{x}{u}\right)\chi_{\delta,x}^c\left(\frac{x}{u}\right)du\right| \le$$
$$\le \left(\frac{1}{e}-2\varepsilon\right)^{n-n_0}\int_A^B u^{mn_0+\alpha-2}e^{-n_0u^m}\left|f\left(\frac{x}{u}\right)\right|du \le$$
$$\le \left(\frac{1}{e}-2\varepsilon\right)^{n-n_0} D\left(m,n_0,\alpha\right)\frac{A^\alpha+B^\alpha}{e^{n_0}}\int_{\frac{a}{B}}^{\frac{b}{A}}\left|f\left(u\right)\right|du = \tag{2.4}$$
$$=\left(\frac{1}{e}-2\varepsilon\right)^{n} M\left(n_0\right) \quad (\text{say}).$$

Since $u^m e^{-u^m}$ is continuous at $u=1$, there exists a $\delta_2>0$ s.t. $u^m e^{-u^m} > \frac{1}{e}-\varepsilon$, for all $u\in\left(1-\delta_2,1+\delta_2\right)$. Therefore

$$\frac{1}{D\left(m,n,\alpha\right)} > \int_{1-\delta_2}^{1+\delta_2} u^{mn+\alpha-2}e^{-nu^m}du > \delta_2\left(\frac{1}{e}-\varepsilon\right)^n \tag{2.5}$$

Thus (2.2), (2.3), (2.4) and (2.5) imply that

$$\left|L_n\left(f\chi_{\delta,x}^c;x\right)\right| \le \frac{M\left(n_0,n_1\right)+M\left(n_0\right)}{\delta_2}\frac{\left(\frac{1}{e}-2\varepsilon\right)^n}{\left(\frac{1}{e}-\varepsilon\right)^n}$$

since $\lim_{n\to\infty} n^k \frac{\left(\frac{1}{e}-2\varepsilon\right)^n}{\left(\frac{1}{e}-\varepsilon\right)^n}=0$ for any $k\in IR^+$ this proves the lemma. □

Next we prove the following basic approximation theorem.

Theorem 1. If $f\in D_\Omega$ and is continuous at a point $x\in IR^+$, then there holds

$$\lim_{n\to\infty} L_n\left(f;x\right)=f\left(x\right) \tag{2.6}$$

further if *f* is continuous on $\langle a,b\rangle$, the convergence (2.6) holds uniformly in $\left[a,b\right]$.

Proof: By continuity of $f\left(u\right)$ at $u=x$, given $\varepsilon>0$ arbitrary we can find a $\delta>0$ such that

$$\left|f\left(u\right)-f\left(x\right)\right| < \frac{\varepsilon}{2}+Ln\left(\left(\left|f\left(u\right)\right|+\left|f\left(x\right)\right|\right)\chi_{\delta,x}^c\left(u\right);x\right) \tag{2.7}$$

where in the case of uniformity δ is independent of $x\in\left[a,b\right]$. In view of (2.7) for all $u\in IR^+$ there holds

$$\left|f(u)-f(x)\right|<\frac{\varepsilon}{2}+\left(\left|f(u)\right|+\left|f(x)\right|\right)\chi^{c}_{\delta,x}(u)$$

Using the linearity, poisitivity and the property that $L_n(1;x)=1$ of L_n from the inequality (2.8) we have

$$\left|L_n(f;x)-f(x)\right|\le\frac{\varepsilon}{2}+L_n\left(\left(\left|f(u)\right|+\left|f(x)\right|\right)\chi^{c}_{\delta,x}(u);x\right)$$

since $\left(\left|f(u)\right|+\left|f(x)\right|\right)\chi^{c}_{\delta,x}(u)\in D_{\Omega}$, using Lemma 1, we can find a n_0 such that

$$L_n\left(\left(\left|f(u)\right|+\left|f(x)\right|\right)\chi^{c}_{\delta,x}(u);x\right)<\frac{\varepsilon}{2}$$

for all $n>n_0$ and ($x\in[a,b]$, in this uniformity case).

Hence $\left|L_n(f;x)-f(x)\right|\le\varepsilon$ for $n>n_0$.

Since $\varepsilon>0$ is arbitrary, the theorem holds. □

3. VORONOVSKAYA THEOREMS

The existence of the third order derivative at the point $u=1$ and the non zero second order derivative at $u=1$ of the function $u^m e^{-u^m}$ ensures that the operators L_n possesses a Voronovskaya-type asymptotic formula. The main result will be followed by the following auxiliary results.

Lemma 2. - If $\delta>0$ is sufficiently small, then the following equalities are true for the operators $L_n(f;x)$.

(i) $$\lim_{n\to\infty} mnD(m,n,\alpha)\int_{1-\delta}^{1+\delta}u^{\alpha+mn+m-3}e^{-(n+1)u^m}\left(1-u^m\right)du=\frac{2-\alpha}{e}$$

(ii) $$\lim_{n\to\infty} m^2nD(m,n,\alpha)\int_{1-\delta}^{1+\delta}u^{\alpha+mn+2m-1}e^{-(n+2)u^m}\left(1-u^m\right)du=\left(\frac{m}{e}\right)^2$$

(iii) $$\lim_{n\to\infty} nD(m,n,\alpha)\int_{1-\delta}^{1+\delta}u^{\alpha+mn-2}e^{-nu^m}\left(\frac{1}{e}-u^m e^{-u^m}\right)du=\frac{1}{2e}$$

Proof: Integrating by parts, taking $u^{\alpha-2}$ as the first function

$$mD(m,n,\alpha)\int_{1-\delta}^{1+\delta}u^{\alpha+mn+m-3}e^{-(n+1)u^m}\left(1-u^m\right)du=$$

$$=D(m,n,\alpha)\left[\frac{u^{\alpha+mn+m-3}e^{-(n+1)u^m}}{n+1}\right]_{1-\delta}^{1+\delta}-\frac{(\alpha+2)}{(n+1)}\int_{1-\delta}^{1+\delta}u^{\alpha+mn+m-3}e^{-(n+1)u^m}\left(1-u^m\right)du$$

for a given $\varepsilon>0$ we can find a $\delta_1\left(0<\delta_1<\delta\right)$ such that

$$\left(\frac{1}{e}-\varepsilon\right)D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1}u^{\alpha+mn-2}e^{-nu^m}du \le$$

$$\le D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1}u^{\alpha+mn+m-3}e^{-(n+1)u^m}du \le$$

$$\le\left(\frac{1}{e}+\varepsilon\right)D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1}u^{\alpha+mn-2}e^{-nu^m}du$$

Applying Theorem 1, we have

$$\lim_{n\to\infty} D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1}u^{\alpha+mn-2}e^{-nu^m}du = 1$$

Hence if n is sufficiently large say $n > n_0$,

$$1-\varepsilon \le D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1}u^{\alpha+mn-2}e^{-nu^m}du \le 1+\varepsilon$$

therefore if $n > n_0$

$$(1-\varepsilon)\left(\frac{1}{e}-\varepsilon\right) \le D(m,n,\alpha)\int_{1-\delta_1}^{1+\delta_1}u^{\alpha+mn+m-3}e^{-(n+1)u^m}du \le (1+\varepsilon)\left(\frac{1}{e}+\varepsilon\right)$$

Let $\left\|\chi_{\delta,1}^{c}u^m e^{-u^m}\right\|_\infty = \left(\frac{1}{e}-2\mu\right)$. Since the function $u^m e^{-u^m}$ attains maximum at $u=1$, we have $\mu > 0$. Also by the continuity of the above function, there exists a $\delta_2\left(0<\delta_2<\delta\right)$ such that

$$\inf_{|u-1|<\delta_2} u^m e^{-u^m} \ge \left(\frac{1}{e}-\mu\right).$$

Hence $D^{-1}(m,n,\alpha) \ge \delta_2\left(\frac{1}{e}-\mu\right)^n$ and therefore if n is sufficiently large

$$\left|D(m,n,\alpha)\left[\frac{u^{\alpha+(n+1)m-2e}e^{-(n+1)u^m}}{n+1}\right]_{1-\delta}^{1+\delta}\right| \le \frac{\varepsilon}{n},\ n > n_1 \text{ (say).}$$

In view of Theorem 1, it is clear that there exists a n_2 such that

$$D(m,n,\alpha)\int_{(1-\delta,1+\delta)/(1-\delta_1,1+\delta_1)}u^{\alpha+mn-2}e^{-nu^m}\chi_{\delta_1,1}^{c}(u)du \le \varepsilon,$$

$n > n_2$ (say).

Making use of the above estimates and the fact that ε is arbitrary, we have (i). □

Proof: (ii) The proof uses similar analysis and the fact that

$$\lim_{n\to\infty} \frac{D(m,n,\alpha)}{D(m,n+1,\alpha)} = e^{-1}$$

Therefore we leave the proof. □

Proof: (iii) Given an arbitrary $\varepsilon > 0$, there exists a $\delta_0 \left(0 < \delta_0 < \frac{1}{1+\delta}\right)$ such that

$$-\frac{1}{2}(1-\varepsilon)u^{2(m-1)}e^{1-2u^m}\left(1-u^m\right)^2 \le u^m e^{-u^m} - e^{-1} \le$$
$$\le -\frac{(1+\varepsilon)u^{2(m-1)}}{2}e^{1-2u^m}\left(1-u^m\right)^2,$$
$$\left|u^{-1}-1\right| < \delta_0$$

Now, using the arguments given in the proof of part 1, the proof easily follows. This completes the proof of the lemma. □

The main results of this section are given in

Theorem 2. If $f \in D_\Omega$, and at a certain point $x \in IR^+$, f'' exists, then there holds

$$L_n(f;x) - f(x) = \frac{xf'(x)[3-2\alpha+m]}{2nm^2} + \frac{x^2 f''(x)}{2nm^2} + o\left(\frac{1}{n}\right) \text{ as } n\to\infty \qquad (3.1)$$

Further, if f'' exists and is continuous on $\langle a,b\rangle$, then (3.1) holds uniformly on $[a,b]$.

Proof: Using L.Hospital's rule we have

$$\lim_{u\to 1} \frac{f\left(\frac{x}{u}\right) - f(x) - \frac{xf'(x)}{m^2}\left[mu^{m-1}\left(1-u^m\right) + (m-3)u^m e^{1-u^m} + (3-m)\right]}{e^{-1} - u^m e^{-u^m}} +$$

$$+\left[\frac{\frac{\frac{2xf'(x) + x^2 f''(x)\left(u^m e^{1-u^m}\right)}{m^2}}{e^2}}{e^{-1} - u^m e^{-u^m}}\right] = 0$$

Hence, given an arbitrary $\varepsilon > 0$ there exists a $\delta > 0$ such that if *u* satisfies $\left|\frac{x}{u} - x\right| < \delta$, there holds

$$f\left(\frac{x}{u}\right) - f(x) - \frac{xf'(x)}{m^2}\left[mu^{m-1}e^{1-u^m} + (m-3)u^m e^{1-u^m} + (3-m)\right] +$$
$$+\left\{2xf'(x) + x^2 f''(x)\left(u^m e^{1-u^m}\right)\right\} / \frac{m^2}{e^2} \le \varepsilon\left(e^{-1} - u^m e^{-u^m}\right).$$

Moreover, it is easily seen that in the uniformity case the above δ can be chosen independent of $x \in [a,b]$. Multiplying the inequality by $u^{\alpha+mn-2}e^{-nu^m}nD(m,n,\alpha)$ and integrating between the limits $(1-\delta, 1+\delta)$ and making use of Lemma 2, we have
which holds uniformly in $x \in [a,b]$, in the uniformity case.

Hence

$$-\frac{\varepsilon}{2e} \le \limsup\left[L_n(f;x) - f(x)\right] + \frac{xf(x)(\alpha-2)}{m^2} + \frac{xf'(x)(3-m)}{2m^2} - \frac{\left[2xf'(x) + x^2 f''(x)\right]}{2m^2} \le \frac{\varepsilon}{2e}$$

In view of the fact that $\varepsilon > 0$ arbitrary, the result follows. □

Corollary 1. Choosing $m = 1$, $\alpha = 2$, we obtain the Voronovskaya formula for the Gamma-Operators of Muller.

$$G_n(f;x) - f(x) = \frac{x^2 f''(x)}{2n} + o\left(\frac{1}{n}\right), \quad n \to \infty .$$

Corollary 2. Taking $m = -1$ and $\alpha = 1$ we have the following Voronovskaya formula for the operators S_n^1.

$$S_n^1(f;x) - f(x) = \frac{x^2 f''(x)}{2n} + o\left(\frac{1}{n}\right), \quad n \to \infty$$

Corollary 3. With $m = -1$ and $\alpha = 0$ we have the Voronovskaya formula for the operators $L_{k,t}$.

$$L_{k,t}(f;x) - f(x) = \frac{xf'(x)}{k} + \frac{x^2 f''(x)}{2k} + o\left(\frac{1}{k}\right), \quad k \to \infty .$$

4. ERROR ESTIMATES

In the previous section, we obtained a precise formula giving the rate of convergence of $L_n(f; x)$ to $f(x)$. The assumption on f has been the existence of its second order derivatives.

If f is only assumed to be continuous, the following theorem gives an estimate of error $\left|L_n(f;x)-f(x)\right|$ in terms of the modulus of continuity of *f*.

Theorem 3. For the operators $L_n(f;x)$ there holds

$$\left|L_n(f;x)-f(x)\right|\le\omega_f\left(n^{-\frac{1}{2}}\right)\left[1+\min\left(x^2\left\{\frac{1}{m^2}+o\left(\frac{1}{n}\right)\right\},x\left\{\frac{1}{m^2}+o(1)\right\}^{\frac{1}{2}}\right)\right],\qquad(4.1)$$

$x\in IR^+$, $n\to\infty$ where ω_f denotes the modulus of continuity of *f* and $o\left(\frac{1}{n}\right)$ are independent of x.

Proof: Using (4.1), we have

$$L_n\left((u-x)^2;x\right)=x^2\left[\frac{1}{nm^2}+o\left(\frac{1}{n}\right)\right],\ n\to\infty\qquad(4.2)$$

By elementary properties of modulus of continuity,

$$\left|f(u)-f(x)\right|\le\omega_f\left(n^{-\frac{1}{2}}\right)\left[1+n^{\frac{1}{2}}\left|u-x\right|\right]\qquad(4.3)$$

and also

$$\left|f(u)-f(x)\right|\le\omega_f\left(n^{-\frac{1}{2}}\right)\left[1+n(u-x)^2\right]\qquad(4.4)$$

For all $x,u\in IR^+$ by Schwartz.s inequality (4.2) implies

$$L_n\left(\left|u-x\right|;x\right)\le\frac{x}{n^{\frac{1}{2}}}\left\{\frac{1}{m^2}+o(1)\right\}^{\frac{1}{2}},\ n\to\infty\qquad(4.5)$$

making use of the linearity and positivity of the operators L_n, (4.1) follows from (4.2)-(4.5). □

For functions which are continuously differentiable the error estimate (4.1) is rather conservative and better estimate is as follows.

Theorem 4. If $f'(x)$ exists and is uniformly continuous on IR^+ there holds,

$$\begin{aligned}\left|L_n(f;x)-f(x)\right|\le&\frac{x\left|f'(x)\right|e^2}{m^4}\left[2(3-\alpha)\frac{m^2}{e}+\frac{m^3-3m^2}{e}+o(1)\right]+\\&+\omega_f\left(n^{-\frac{1}{2}}\right)\left[\frac{x}{n^{\frac{1}{2}}}\left(\left\{\frac{1}{m^2}\right\}^{\frac{1}{2}}+o(1)\right)+\frac{x^2}{2n^{\frac{1}{2}}}\left(\left\{\frac{1}{m^2}\right\}^{\frac{1}{2}}+o(1)\right)\right]\end{aligned}\qquad(4.6)$$

$x \in IR^+$, $n \to \infty$, where ω_f denotes the modulus of continuity of *f*.

Proof: We have,

$$\left| f(u) - f(x) - (u-x) f'(x) \right| \le \left| \int_x^u \left(f'(u) - f'(x) \right) du \right| \le \left| \int_x^u \omega_{f'} \left(|u-x| \right) du \right| \le$$
$$\le \left| \int_x^u \omega_{f'} \left(n^{-\frac{1}{2}} \right) \left(1 + n^{\frac{1}{2}} |u-x| \right) du \right| = \tag{4.7}$$
$$= \omega_{f'} \left(n^{-\frac{1}{2}} \right) \left\{ |u-x| + \frac{1}{2} n^{\frac{1}{2}} (u-x)^2 \right\}$$

Since by Theorem 2 we have

$$L_n\left((u-x);x\right) = \frac{x}{2nm^2}(3 - 2\alpha + m) + o\left(n^{-1}\right)$$

The inequality (4.6) follows by operating (4.7) by L_n and making use of (4.2), (4.4) and (4.8). □

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