ORIGINAL PAPER

A TRIGONOMETRIC METHOD OF SOLVING THE HADWIGER-FINSLER INEQUALITY IN ACUTE TRIANGLE

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Manuscript received: 05.03.2012; Accepted paper: 13.07.2012;
Published online: 01.12.2012.

Abstract. *In the paper [1] the authors proposed the following inequality:*
In any acute triangle are true the following inequality:

$$a^2+b^2+c^2 \le 4\sqrt{3}s+\frac{2-\sqrt{3}}{3-2\sqrt{2}\left[(a-b)^2+(b-c)^2+(c-a)^2\right]} \tag{1}$$

The aim of this paper is to give a trigonometric proof of the inequality (1)
Keywords: *integral equation, selfadjoint operator, strongly positive linear operator.*

1. MAIN RESULTS

In any triangle ABC we shall denote by $a=\left\|\overrightarrow{BC}\right\|$, $b=\left\|\overrightarrow{AC}\right\|$, $c=\left\|\overrightarrow{AB}\right\|$, $p=\frac{a+b+c}{2}$, R the radius of circumcircle and r the radius of incircle.

We shall consider $A \ge B \ge C$ and we shall denote: $\lambda=\frac{2-\sqrt{3}}{3-2\sqrt{2}}>1$. We have $\frac{\pi}{3} \le A \le \frac{\pi}{2}$.

Inequality (1) may be written in an equivalent forms as:

$$\left(a^2+b^2+c^2\right)(1-2\lambda)+2\lambda(ab+bc+ca)-4\sqrt{3}s \le 0 \tag{2}$$

and because $a=2R\sin A$, $b=2R\sin B$, $c=2R\sin C$, $S=2R^2\sin C$ it follows that inequality (2) is equivalent with:

$$\begin{aligned} f(A,B,C)=(1-2\lambda)\left(\sin^2 A+\sin^2 B+\sin^2 C\right)+ \\ +2\lambda(\sin A\sin B+\sin B\sin C+\sin C\sin A)-2\sqrt{3}\sin A\sin B\sin C \le 0 \end{aligned} \tag{3}$$

In order to prove the inequality (3) it will be sufficient to prove that :

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$$f\left(A,B,C\right)\le f\left(A,\frac{B+C}{2},\frac{B+C}{2}\right) \qquad (4)$$

$$f\left(A,\frac{B+C}{2},\frac{B+C}{2}\right)\le 0 \qquad (5)$$

The inequality (4) may be written in an equivalent form as:

$$\left(1-2\lambda\right)\left(\sin^2 B+\sin^2 C-2\cos^2\frac{A}{2}\right)+2\lambda\sin A\left(\sin B+\sin C\right)+2\lambda\sin B\sin C-$$
$$-4\lambda\sin A\cos\frac{A}{2}-2\lambda\cos^2\frac{A}{2}-2\sqrt{3}\sin A\sin B\sin C+2\sqrt{3}\sin A\cos^2\frac{A}{2}\le 0$$

or :

$$\left(4-8\lambda\right)\cos^2\frac{A}{2}\left(\cos^2\frac{B-C}{2}-1\right)+\left(2-4\lambda\right)\cos A+2-4\lambda-\left(2-4\lambda\right)\left(\cos^2\frac{B-C}{2}-1\right)-$$
$$-1+2\lambda-1+2\lambda-\left(2-4\lambda\right)\cos A+4\lambda\sin A\cos\frac{A}{2}\left(\cos\frac{B-C}{2}-1\right)+4\lambda\sin A\cos\frac{A}{2}+$$
$$+2\lambda\left(\cos^2\frac{B-C}{2}-1\right)+\lambda+\lambda\cos A-4\lambda\sin A\cos\frac{A}{2}-\lambda\cos A-\lambda-$$
$$-2\sqrt{3}\sin A\left(\cos^2\frac{B-C}{2}-1\right)-\sqrt{3}\sin A-\sqrt{3}\sin A\cos A+\sqrt{3}\sin A\cos A+\sqrt{3}\sin A\le 0$$

or

$$\left(1-\cos[\![(B-C)/2]\!]\right)\Big[\left(4-8\lambda\right)[\![\cos]\!]^2 A/2\left(\cos\left(B-C\right)/2+1\right)+$$
$$+\left(4\lambda-2\right)\left(\cos\left(B-C\right)/2+1\right)+4\lambda\sin A\cos A/2+2\lambda\left(\cos\left(B-C\right)/2+1\right) \qquad (6)$$
$$-2\sqrt{3}\sin A\left(\cos\left(B-C\right)/2+1\right)\Big]\ge 0$$

In order to prove the inequality (6) it will be sufficient to prove that:

$$\frac{\left(\frac{\cos\left(B-C\right)}{2}+1\right)\left[\left(1-2\lambda\right)\cos A-\sqrt{3}\sin A+\lambda\right]+2\lambda\sin A\cos A}{2}\ge 0$$

We shall consider the function: $f:\left[-\frac{\pi}{2},\frac{\pi}{2}\right]\to R$

$$f\left(A\right)=\frac{\left(\frac{\cos\left(B-C\right)}{2}+1\right)\left[\left(1-2\lambda\right)\cos A-\sqrt{3}\sin A+\lambda\right]+2\lambda\sin A\cos A}{2}$$

Calculating the derivate for *f(A)* we get :

$$f'(A)=\frac{\dfrac{\left[(2\lambda-1)\sin A-\sqrt{3}\cos A\right]\left(\dfrac{\cos(B-C)}{2}+1\right)+2\lambda\cos A\cos A}{2}-\lambda\sin A\sin A}{2} \quad (7)$$

We prove that $f'(A)\geq 0,\ (\forall)A\in\left[\frac{\pi}{3},\frac{\pi}{2}\right]$

Because $A\geq\frac{\pi}{2}$ we have $\lambda\mathrm{tg}A\geq\lambda\sqrt{3}\geq\sqrt{3}$ or $\lambda\sin A-\sqrt{3}\cos A\geq 0$

We shall obtain :

$$(2\lambda-1)\sin A-\sqrt{3}\cos A=(\lambda-1)\sin A+\lambda\sin A-\sqrt{3}\cos A\geq 0 \quad (8)$$

From (7) and (8) we shall obtain :

$$f'(A)\geq\frac{\dfrac{(2\lambda-1)\sin A-\sqrt{3}\cos A+2\lambda\cos A\cos A}{2}-\lambda\sin A\sin A}{2}= \quad (9)$$
$$=\sin A\left(2\lambda-1-\frac{\lambda\sin A}{2}\right)+\cos A\left(\frac{2\lambda\cos A}{2}-\sqrt{3}\right)$$

But we have: $2\lambda-1-\frac{\lambda\sin A}{2}=\lambda-1+\lambda\left(1-\frac{\sin A}{2}\right)>0$ (10)

Because $\frac{A}{2}\leq\frac{\pi}{4}$ we have $\frac{\cos A}{2}\geq\frac{1}{\sqrt{2}}$ or $\frac{2\lambda\cos A}{2}\geq\sqrt{2}\lambda>\sqrt{3}$ and in an equivalent form as: $\frac{2\lambda\cos A}{2}-\sqrt{3}>0$ (11)

From (9) (10) (11) we shall obtain $f'(A)>0,\ (\forall)A\in\left[\frac{\pi}{3},\frac{\pi}{2}\right]$ from where f is an increasing function on $\left[\frac{\pi}{3},\frac{\pi}{2}\right]$ who is equivalent with :

$$f(A)\geq f\left(\frac{\pi}{3}\right)=\frac{3\lambda-\dfrac{2\cos(B-C)}{2}-2}{2}\geq\frac{3\lambda-4}{2}>0$$

and we have the inequality (4)

In order to prove the inequality (5) it will be sufficient $f(A,B,B)\leq 0$ (12) we shall denote: $y=\frac{\sin B}{\sin A}\leq 1$

From $A = B$ we shall obtain equality.

We study the case $A\neq B$

We have $\sin^2 B + \sin^2 B > \sin^2 A$ who implies: $y = \dfrac{\sin B}{\sin A} > \dfrac{1}{\sqrt{2}}$

Inequality (12) will be written in an equivalent form as:

$$\frac{\sin^2 A + 2\sin^2 B - \sin A\sqrt{12\sin^2 B - 3\sin^2 A}}{2(\sin A - \sin B)^2} \le \lambda \text{ or } \frac{2\lambda^2 + 1 - \sqrt{12\lambda^2 - 3}}{2(\lambda - 1)^2} \le \lambda$$

We shall consider the function:

$$u : \left[\frac{1}{\sqrt{2,1}}\right) \to R,\ V : \left[\frac{1}{\sqrt{2,1}}\right) \to R;$$

$$u(y) = \frac{2y^2 + 1 - \sqrt{12y^2 - 3}}{2(y-1)^2},$$

$$V(y) = 2y^2 + 1 - \sqrt{12y^2 - 3} \text{ and } V'(y) = 4y - \frac{12y}{\sqrt{12y^2 - 3}} < 0 \text{ because } y < 1$$

Because u is an decreasing function we have $u(y) < u\left(\dfrac{1}{\sqrt{2}}\right)$. From (4) and (5) we shall obtain $f(A,B,C) \le 0$.

REFERENCE

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