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ORIGINAL PAPER

NEW GENERALIZATIONS AND NEW APPLICATIONS FOR NESBITT'S INEQUALITY

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Abstract. *This paper presents new generalizations and new refinements for Nesbitt's inequality (other that those of [1] and [2]).*

Keywords: *Nesbitt's inequality, Jensen's inequality, P.L. Chebyshev's inequality.*

1. INTRODUCTION

The Nesbitt's inequality (see e.g. [1], and [3]) is

$$\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\geq\frac{3}{2},\quad \text{for any } x,y,z\in\mathbf{R}_+^*$$

In this paper we prove a generalization and refinement for *Nesbitt's* inequality and by particularization we obtain others generalizations and refinements of *Nesbitt's* inequality.

2. A GENERALIZATION OF NESBITT'S INEQUALITY

Theorem 2.1. *If* $n\in N^*-\{1\}$, $a\in R_+$, $b,c,d,x_k\in R_+^*$, $\forall k=\overline{1,n}$, $X_n=\sum_{k=1}^n x_k$, $m,p,r,s\in[1,\infty)$, *such that* $cX_n^r>d\max_{1\leq k\leq n}x_k^r$, *then*

$$\sum_{k=1}^n\frac{\left(aX_n^m+bx_k^m\right)^p}{\left(cX_n^r-dx_k^r\right)^s}\geq\frac{\left(an^m+b\right)^p}{\left(cn^r-d\right)^s}n^{rs-mp+1}X_n^{mp-rs}\tag{1}$$

Proof. If we denote $y_k=\frac{x_k}{X_n}$, $\forall k=\overline{1,n}$, then $Y_n=\sum_{k=1}^n y_k=1$, and (1) becomes

$$U_n=\sum_{k=1}^n\frac{\left(aX_n^m+bx_k^m\right)^p}{\left(cX_n^r-dx_k^r\right)^s}=X_n^{mp-rs}\sum_{k=1}^n\frac{\left(a+by_k^m\right)^p}{\left(c-dy_k^r\right)^s}\geq\frac{\left(an^m+b\right)^p}{\left(cn^r-d\right)^s}n^{rs-mp+1}X_n^{mp-rs}\tag{2}$$

We prove (2) by considering the functions

$$f,g,h:\left(0,\left(\frac{c}{d}\right)^{\frac{1}{r}}\right)\to R_+^*,\ f(y)=\left(a+by^m\right)^p,\ g(y)=\left(c-dy^r\right)^{-s},\ h=fg\,.$$

We have

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$$f'(y) = bmp\left(a+by^m\right)^{p-1} y^{m-1} > 0,\ g'(y) = drs\left(c-dy^r\right)^{-(s+1)} y^{r-1} > 0,\ \forall y \in \left(0,\left(\frac{c}{d}\right)^{\frac{1}{r}}\right), \text{ and}$$

$$f''(y) = bmp\left(a+by^m\right)^{p-2} y^{m-2}\left((m-1)a + b(mp-1)y^m\right) > 0,$$

$$g''(y) = drs\left(c-dy^r\right)^{-(s+2)} y^{r-2}\left((r-1)c + d(rs+1)y^r\right) > 0,\ \forall y \in \left(0,\left(\frac{c}{d}\right)^{\frac{1}{r}}\right).$$

Because, $h = fg$, we have $h'' = f''g + 2f'g' + fg''$ and then

$$h''(y) = f''(y)g(y) + 2f'(y)g'(y) + f(y)g''(y) > 0,\ \forall y \in \left(0,\left(\frac{c}{d}\right)^{\frac{1}{r}}\right),$$

So that, h is convex on $\left(0,\left(\frac{c}{d}\right)^{\frac{1}{r}}\right)$ and by *Jensen' s* inequality we deduce that

$$\sum_{k=1}^{n} h(y_k) = \sum_{k=1}^{n} \frac{\left(a+by_k^m\right)^p}{\left(c-dy_k^r\right)^s} \ge nh\left(\frac{1}{n}\sum_{k=1}^{n} y_k\right) = nh\left(\frac{1}{n}Y_n\right) = nh\left(\frac{1}{n}\right) = n \cdot \frac{\left(a+\frac{b}{n^m}\right)^p}{\left(c-\frac{d}{n^r}\right)^s} =$$

$$= \frac{\left(an^m + b\right)^p}{\left(cn^r - d\right)^s} n^{rs-mp+1} \tag{3}$$

By (2) and (3) we obtain that

$$U_n \ge \frac{\left(an^m + b\right)^p}{\left(cn^r - d\right)^s} n^{rs-mp+1} X_n^{mp-rs},$$

so (1) is proved.

3. A REFINEMENT OF NEBITT'S INEQUALITY

Theorem 3.1. *If* $n \in N^* - \{1\}$, $a \in R_+$, $b,c,d,x_k \in R_+^*$, $\forall k = \overline{1,n}$, $X_n = \sum_{k=1}^{n} x_k$, $m,p,r,s \in [1,\infty)$*, such that* $cX_n^r > d \max_{1\le k\le n} x_k^r$*, then*

$$\sum_{k=1}^{n} \frac{\left(aX_n^m + bx_k^m\right)^p}{\left(cX_n^r - dx_k^r\right)^s} \ge \frac{1}{n}\left(\sum_{k=1}^{n}\left(aX_n^m + bx_k^m\right)^p\right)\sum_{k=1}^{n}\frac{1}{\left(cX_n^r - dx_k^r\right)^s} \ge$$

$$\ge \frac{\left(anX_n^m + b\sum_{k=1}^{n} x_k^m\right)^p}{\left(cnX_n^r - d\sum_{k=1}^{n} x_k^r\right)^s} n^{s-p+1} \ge \frac{\left(an^m + b\right)^p}{\left(cn^r - d\right)^s} n^{rs-mp+1} X_n^{mp-rs} \tag{4}$$

Proof: WLOG we can assume that $x_1 \le x_2 \le ... \le x_n$ and then

$$\left(aX_n^m + bx_1^m\right)^p \le \left(aX_n^m + bx_2^m\right)^p \le \ldots \le \left(aX_n^m + bx_n^m\right)^p \tag{5}$$

and

$$\frac{1}{\left(cX_n^r - dx_1^r\right)^s} \le \frac{1}{\left(cX_n^r - dx_2^r\right)^s} \le \ldots \le \frac{1}{\left(cX_n^r - dx_n^r\right)^s} \tag{6}$$

By* P.L. Chebyshev's *inequality we have

$$U_n = \sum_{k=1}^{n} \frac{\left(aX_n^m + bx_k^m\right)^p}{\left(cX_n^r - dx_k^r\right)^s} \ge \frac{1}{n}\left(\sum_{k=1}^{n}\left(aX_n^m + bx_k^m\right)^p\right)\sum_{k=1}^{n}\frac{1}{\left(cX_n^r - dx_k^r\right)^s} \tag{7}$$

Since the function $u : R_+^* \to R_+^*$, $u(t) = t^p$, $p \ge 1$ is convex on R_+^* we deduce that

$$\sum_{k=1}^{n}\left(aX_n^m + bx_k^m\right)^p \ge n\left(\frac{1}{n}\sum_{k=1}^{n}\left(aX_n^m + bx_k^m\right)\right)^p = \frac{1}{n^{p-1}}\left(anX_n^m + b\sum_{k=1}^{n}x_k^m\right)^p \tag{8}$$

By J. Radon's inequality we obtain that

$$\sum_{k=1}^{n}\frac{1}{\left(cX_n^r - dx_k^r\right)^s} \ge \frac{n^{s+1}}{\left(\sum_{k=1}^{n}\left(cX_n^r - dx_k^r\right)\right)^s} = \frac{n^{s+1}}{\left(cnX_n^r - d\sum_{k=1}^{n}x_k^r\right)^s} \tag{9}$$

By (7), (8) and (9) yields that

$$U_n \ge \frac{\left(anX_n^m + b\sum_{k=1}^{n}x_k^m\right)^p}{\left(cnX_n^r - d\sum_{k=1}^{n}x_k^r\right)^s} n^{s-p+1} \tag{10}$$

Since the functions $v, w : R_+^* \to R_+^*$, $v(t) = t^m$, $w(t) = t^r$, $m, r \ge 1$ ar convex on R_+^* we have

$$\sum_{k=1}^{n} x_k^m \ge \frac{X_n^m}{n^{m-1}} \tag{11}$$

respectively

$$\sum_{k=1}^{n} x_k^r \ge \frac{X_n^r}{n^{r-1}} \tag{12}$$

By (11) and (12) the inequality (10) becomes

$$U_n \ge \frac{\left(anX_n^m + \frac{b}{n^{m-1}}X_n^m\right)^p}{\left(cnX_n^r - \frac{d}{n^{r-1}}X_n^r\right)^s} n^{s-p+1} = \frac{\left(an^m + b\right)^p}{\left(cn^r - d\right)^s} n^{rs-mp+1} X_n^{mp-rs} \tag{13}$$

Therefore

$$U_n \ge \frac{\left(anX_n^m + b\sum_{k=1}^{n}x_k^m\right)^p}{\left(cnX_n^r - d\sum_{k=1}^{n}x_k^r\right)^s} n^{s-p+1} \ge \frac{\left(an^m + b\right)^p}{\left(cn^r - d\right)^s} n^{rs-mp+1} X_n^{mp-rs} \tag{14}$$

4. SOME CONSEQUENCES

Corollary 4.1. *If* $n \in N^* - \{1\},\ a \in R_+,\ b,\ c,\ d,\ x_k \in R_+^*,\ \forall k = \overline{1,n},\ X_n = \sum_{k=1}^{n} x_k$, $\alpha \in [1, \infty)$*, such that* $cX_n > d \max_{1 \le k \le n} x_k$*, then*

$$\sum_{k=1}^{n} \frac{(aX_n + bx_k)^\alpha}{cX_n - dx_k} \ge \frac{(an+b)^\alpha}{cn-d} n^{2-\alpha} X_n^{\alpha-1} \tag{15}$$

Proof: In theorem 2.1., we take $m = r = s = 1,\ p = \alpha$.

Corollary 4.2. *If* $n \in N^* - \{1\},\ x_k \in R_+^*,\ \forall k = \overline{1,n},\ X_n = \sum_{k=1}^{n} x_k,\ \alpha \in [1, \infty)$*, then*

$$\sum_{k=1}^{n} \frac{x_k^\alpha}{X_n - x_k} \ge \frac{n^{2-\alpha}}{n-1} X_n^{\alpha-1} \tag{16}$$

Proof: In corollary 4.1. we take $a = 0, b = c = d = 1$ and we obtain (16), i.e. the corollary 2.1. from [2].

Corollary 4.3. *If* $n \in N^* - \{1\},\ x_k \in R_+^*,\ \forall k = \overline{1,n},\ X_n = \sum_{k=1}^{n} x_k$*, then*

$$\sum_{k=1}^{n} \frac{x_k}{X_n - x_k} \ge \frac{n}{n-1} \tag{17}$$

Proof: In corollary 4.2. we take $\alpha = 1$ and we obtain the corollary 2.2. from [2] .

Remark 4.1. *For* $n = 3$ *by corollary 4.3. we obtain the "classical" Nesbitt's inequality.*

$$\frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_1} + \frac{x_3}{x_1 + x_2} \ge \frac{3}{2},\ \forall x_1, x_2, x_3 \in R_+^* \tag{18}$$

Corollary 4.5. *If* $n \in N^* - \{1\},\ a \in R_+,\ b, c, d, x_k \in R_+^*,\ \forall k = \overline{1,n},\ X_n = \sum_{k=1}^{n} x_k$, $cX_n > d \max_{1 \le k \le n} x_k,\ \alpha \in R_+$*, then*

$$\sum_{k=1}^{n} \frac{(aX_n + bx_k)^{\alpha+1}}{cX_n - dx_k} \ge \frac{\left(anX_n + b\sum_{k=1}^{n} x_k\right)^{\alpha+1}}{cnX_n - d\sum_{k=1}^{n} x_k} n^{1-\alpha} \ge \frac{(an+b)^{\alpha+1}}{cn-d} n^{1-\alpha} X_n^\alpha \tag{19}$$

Proof: In theorem 3.1. we take $p = \alpha + 1,\ m = r = s = 1$.

Corollary 4.6. *If* $n \in N^* - \{1\},\ c, d, x_k \in R_+^*,\ \forall k = \overline{1,n},\ X_n = \sum_{k=1}^{n} x_k,\ cX_n > d \max_{1 \le k \le n} x_k$, $\alpha \in R_+$*, then*

$$\sum_{k=1}^{n} \frac{x_k^{\alpha+1}}{cX_n - dx_k} \ge \frac{\sum_{k=1}^{n} x_k^{\alpha+1}}{n} \sum_{k=1}^{n} \frac{1}{cX_n - dx_k} \ge \frac{n^{1-\alpha}}{cn-d} X_n^\alpha \tag{20}$$

Proof: In (4) we take $a = 0,\ b = 1,\ m = r = s = 1,\ p = \alpha + 1$, and we deduce that

$$\sum_{k=1}^{n} \frac{x_k^{\alpha+1}}{cX_n - dx_k} \geq \frac{1}{n} X_n^{\alpha+1} \cdot \sum_{k=1}^{n} \frac{1}{cX_n - dx_k} \geq \frac{n^{1-\alpha}}{cn - d} X_n^{\alpha},$$

i.e. the theorem 4.1. from [2].

Corollary 4.7. *If* $n \in N^* - \{1\},\ x_k \in R_+^*,\ \forall k = \overline{1,n},\ X_n = \sum_{k=1}^{n} x_k,\ \alpha \in R_+$, *then*

$$\sum_{k=1}^{n} \frac{x_k^{\alpha+1}}{X_n - x_k} \geq \frac{1}{n}\left(\sum_{k=1}^{n} x_k^{\alpha+1}\right)\sum_{k=1}^{n} \frac{1}{X_n - x_k} \geq \frac{n^{1-\alpha}}{n-1} X_n^{\alpha} \tag{21}$$

Proof: In corollary 4.6. we take $c = d = 1$. In this way we reobtain the corollary 4.1. from [2].

Remark 4.2. *The inequality (21) is a refinement of the inequality 2.2. from [2], thus that (21) is a generalization and a refinement of Nesbitt's inequality.*

Corollary 4.8. *If* $x, y, z \in R_+^*$, *then*

$$\begin{aligned} &\frac{x^{\alpha+1}}{c(y+z) - (d-c)x} + \frac{y^{\alpha+1}}{c(z+x) - (d-c)y} + \frac{z^{\alpha+1}}{x(x+y) - (d-c)z} \geq \\ &\geq \frac{1}{3}(x+y+z)^{\alpha+1}\left(\frac{1}{c(y+z) - (d-c)x} + \frac{1}{c(z+x) - (d-c)y} + \frac{1}{c(x+y) - (d-c)z}\right) \geq \\ &\geq \frac{3^{1-\alpha}}{3c - d} \end{aligned} \tag{22}$$

Proof: In (20) we take $n = 3$.

Corollary 4.9. *If* $x, y, z \in R_+^*$, *then*

$$\frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y} \geq \frac{1}{3}(x+y+z)\left(\frac{1}{y+z} + \frac{1}{z+x} + \frac{1}{x+y}\right) \geq \frac{3}{2} \tag{23}$$

Proof: In (22) we take $\alpha = 0$, i.e. we reobtain the corollary 4.2. from [2].

Remark 4.3. *The inequality (23) is a refinement of Nesbitt's inequality.*

5. SOME APPLICATIONS

Application 5.1. *If* $n \in N^* - \{1\},\ a \in R^+,\ b, c, d, a_k \in R_+^*,\ \forall k = \overline{1,n},\ H_n = \sum_{k=1}^{n} \frac{1}{a_k}$, $m, p, r, s \in [1, \infty)$, *and* $cH_n^r > d \max_{1 \leq k \leq n} \frac{1}{a_k}$, *then*

$$\sum_{k=1}^{n} \frac{\left(aH_n^m + \frac{b}{a_k^m}\right)^p}{\left(cH_n^r - \frac{d}{a_k^r}\right)^s} \geq \frac{(an^m + b)^p}{(cn^r - d)^s} n^{rs - mp + 1} H_n^{mp - rs} \tag{24}$$

Proof: In theorem 2.1. we take $x_k = \frac{1}{a_k}$, $k = \overline{1,n}$, and we obtain (24). If $a_1 a_2 ... a_n = 1$, then $H_n \ge n \cdot \sqrt[n]{\frac{1}{a_1 a_2 ... a_n}} = n$, and (24) becomes

$$\sum_{k=1}^{n} \frac{\left(aH_n^m + \frac{b}{a_k^m}\right)^p}{\left(cH_n^r - \frac{d}{a_k^r}\right)^s} \ge \frac{(an^m + b)^p}{(cn^r - d)^s} n^{rs-mp+1} n^{mp-rs} = \frac{(an^m + b)^p}{(cn^r - d)^s} n \qquad (25)$$

If we consider $n = 3$, then by (25) we have

$$\frac{\left(aH_3^m + \frac{b}{a_1^m}\right)^p}{\left(cH_3^r - \frac{d}{a_1^r}\right)^s} + \frac{\left(aH_3^m + \frac{b}{a_2^m}\right)^p}{\left(cH_3^r - \frac{d}{a_2^r}\right)^s} + \frac{\left(aH_3^m + \frac{b}{a_3^m}\right)^p}{\left(cH_3^r - \frac{d}{a_3^r}\right)^s} \ge 3 \cdot \frac{(3^m a + b)^p}{(3^r c - d)^s} \qquad (26)$$

and for $a = 0, b = c = d = 1$, we deduce that

$$\frac{1}{a_1^{mp}\left(\frac{1}{a_2^r} + \frac{1}{a_3^r}\right)^s} + \frac{1}{a_2^{mp}\left(\frac{1}{a_3^r} + \frac{1}{a_1^r}\right)^s} + \frac{1}{a_3^{mp}\left(\frac{1}{a_1^r} + \frac{1}{a_2^r}\right)^s} \ge \frac{3}{(3^r - 1)^s} \Leftrightarrow$$

$$\Leftrightarrow \frac{1}{a_1^{mp+rs}(a_2^r + a_3^r)^s} + \frac{1}{a_2^{mp+rs}(a_3^r + a_1^r)^s} + \frac{1}{a_3^{mp+rs}(a_1^r + a_2^r)^s} \ge \frac{3}{(3^r - 1)^s} \qquad (27)$$

If we take $m = 2,\ p = r = s = 1$, then by (27) we have

$$\frac{1}{a_1^3(a_2 + a_3)} + \frac{1}{a_2^3(a_3 + a_1)} + \frac{1}{a_3^3(a_1 + a_2)} \ge \frac{3}{2} \qquad (28)$$

i.e. we obtain the problem proposed by Russia at 36-th I.M.O., Canada, 1995.

Application 5.2. *If we consider* $a = 0, b = c = d = 1,\ p \in N^*$ *and* $m = r = s = 1$, *then by (1) we have*

$$\sum_{k=1}^{n} \frac{x_k^p}{X_n - x_k} \ge \frac{X_n^{p-1}}{n^{p-2}(n-1)} \qquad (29)$$

i.e. the problem O:1087 from Romanian Mathematical Gazette, no. 5/2005, p. 226 , proposed by *Gh. Ivancev* (Vidin, Bulgaria) and *Lucian Tuțescu* (Craiova, Romania).

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