ORIGINAL PAPER

q-FRACTIONAL INITIAL VALUE PROBLEMS AND q-ANALOGUE OF GENERALIZED MITTAG-LEFFLER FUNCTION

KISHAN SHARMA[1]

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Abstract. *In the present paper, the author introduced q-analogues of generalized Mittag-Leffler function referred by Prabhakar [23]. Results derived in this paper are the extensions of the results derived earlier by Abdeljawad et al. [19]. Some special cases of interest are also discussed.*

Keywords: *Left q-fractional integral, right q-fractional integral, Caputo left and right q-fractional derivatives, generalized Mittag-Leffler function.*

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1. INTRODUCTION

The calculus of the real order derivatives and integrals has become very suitable apparatus in describing and solving a lot of problems in numerous sciences, such as physics, electrochemistry and material science (see, for example Podlubny [9]).Their treatment from the point of view of the q-calculus can additionally open new perspectives as it did, for example, in optimal control problems (see Bangerezako [4]). The fractional q-calculus is the q-extension of the ordinary fractional calculus. The theory of q-calculus operators in recent past have been applied in the areas like ordinary fractional calculus, optimal control problems, solutions of q-difference(differential) and q-integral equations, q-transform analysis etc. Recently, Abu-Risha, Annaby, Ismail and Mansour [13] and Mansour [26] derived the fundamental set of solutions for the homogenous linear sequential fractional q-difference equations with constant coefficients. Fang [10] and Purohit [18] deduced several transformations and summations formulae for the basic hypergeometric functions as the applications of fractional q-differential operator.

For more details one may refer the recent papers [5, 14, 18] on the subject.

Very recently and after the appearance of time scale calculus (see for example [12]), some authors started to pay attention and apply the techniques of time scale to discrete fractional calculus [1-3, 20] benefitting from the results announced before in [11]. All of these results are mainly about fractional calculus on the time scales $T_q = \{q^n : n \in Z\} \cup \{0\}$ and hZ [16]. However, the study of fractional calculus on time scales combining the previously mentioned time scales is still unknown. Counting in this direction and being motivated by all above, in this paper we study Caputo type q-fractional derivatives and use a direct method to derive the solution of a certain linear Caputo q-fractional difference equation by means of a new introduced generalized q-Mittag-Leffler function and some special cases are also discussed.

[1] NRI Institute of Technology and Management, Department of Mathematics, 474001 Gwalior, India.
E-mail: drkishansharma2006@rediffmail.com; drkishan010770@yahoo.com.

For the theory of q-calculus we refer the reader to the survey of [22] and for the basic definitions and results for the q-fractional calculus we refer to [3].

For $0 < q < 1$, let T_q be the time scale:

$T_q = \{q^n : n \in Z\} \cup \{0\}$, where Z is the set of integers.

More generally, if α is a nonnegative real number then we define the time scale $T_q^{\alpha} = \{q^n : n \in Z\} \cup \{0\}$ with $T_q^{0} = T_q$.

For a function $f : T_q \to R$, the nabla q-derivative of f is given by

$$(\nabla_q f)(t) = \frac{f(t) - f(qt)}{(1-q)t}, t \in T_q - \{0\}. \tag{1.1}$$

and

$$(\nabla_q f)(0) = \lim_{t \to 0} (\nabla_q f)(t),$$

where $\nabla_q f \to \frac{d}{dt}$ as $q \to 1$.

The nabla q-integral of a function is defined by

$$\int_0^t f(x) \nabla_q x = t(1-q) \sum_{k=0}^{\infty} q^k f(t q^k), \tag{1.2}$$

For $0 \le a \in T_q$,

$$\int_a^t f(x) \nabla_q x = \int_0^t f(x) \nabla_q x - \int_0^a f(x) \nabla_q x \tag{1.3}$$

On the other hand

$$\int_t^{\infty} f(x) \nabla_q x = t(1-q) \sum_{k=1}^{\infty} q^{-k} f(t q^{-k}), \tag{1.4}$$

For $0 < b < \infty$ in T_q

$$\int_t^b f(x) \nabla_q x = \int_t^{\infty} f(x) \nabla_q x - \int_b^{\infty} f(x) \nabla_q x \tag{1.5}$$

$$\int_0^{\infty} f(x) \nabla_q x = (1-q) \sum_{k=-\infty}^{\infty} q^k f(q^k). \tag{1.6}$$

By the fundamental theorem of q-calculus, we have

$$\nabla_q \int_0^t f(x) \nabla_q x = f(t) \tag{1.7}$$

and if f is continuous at 0, then

$$\int_0^t \nabla_q f(x) \nabla_q x = f(t) - f(0). \tag{1.8}$$

Also the following identity will be helpful

$$\nabla_q \int_a^t f(t,x)\nabla_q x = \int_a^t \nabla_q f(t,x)\nabla_q x + f(qt,t) \tag{1.9}$$

The following will be useful

$$\nabla_q \int_t^b f(t,x)\nabla_q x = \int_{qt}^b \nabla_q f(t,x)\nabla_q x - f(t,t) \tag{1.10}$$

$$\nabla_q (f(t)g(t)) = f(qt)\nabla_q g(t) + (\nabla_q f(t))g(t) \tag{1.11}$$

The q-factorial function for $n \in N$ is defined as

$$(t-x)_q^n = \prod_{j=0}^{n-1}(t-q^j x) \tag{1.12}$$

When α is a non-positive integer, the factorial function is defined as

$$(t-x)_q^\alpha = t^\alpha \prod_{j=0}^{\infty} \frac{1-\frac{x}{t}q^j}{1-\frac{x}{t}q^{j+\alpha}} \tag{1.13}$$

The properties of q-factorial functions, which can be found mainly in [3], in the following lemma:

Lemma 1: For $\alpha, \beta, \gamma \in R$ then

(i) $$(t-x)_q^{\beta+\gamma} = (t-x)_q^\beta (t-q^\beta x)_q^\gamma \tag{1.14}$$

(ii) $$(at-ax)_q^\beta = a^\beta (t-x)_q^\beta \tag{1.15}$$

(iii)The nabla q-derivative of the q-factorial function with respect to *t* is

$$\nabla_q (t-x)_q^\alpha = \frac{1-q^\alpha}{1-q}\ (t-x)_q^{\alpha-1} \tag{1.16}$$

(iv) The nabla q-derivative of the q-factorial function with respect to *x* is

$$\nabla_q (t-x)_q^\alpha = -\frac{1-q^\alpha}{1-q}\ (t-qx)_q^{\alpha-1} \tag{1.17}$$

For the q-gamma function we refer the reader to [3]

$$\Gamma_q(\alpha+1)=\frac{1-q^{\alpha}}{1-q}\Gamma_q(\alpha),\Gamma_q(1)=1,\alpha>0. \tag{1.18}$$

The authors in [3] following [17] defines the left fractional q-integral of order $\alpha(\neq 0,-1,-2,...)$ by

$$_qI^{\alpha}f(t)=\frac{1}{\Gamma_q(\alpha)}\int_0^t(t-qx)_q^{(\alpha-1)}f(x)\nabla_q x \tag{1.19}$$

and it was proved that the left fractional q-integral obeys the identity

$$_qI^{\beta}\,_qI^{\alpha}f(t)=\,_qI^{\alpha+\beta}f(t);\alpha,\beta>0. \tag{1.20}$$

The left q-fractional integral for $0<a\in T_q$ is defined by

$$_qI_a^{\alpha}f(t)=\frac{1}{\Gamma_q(\alpha)}\int_a^t(t-qx)_q^{(\alpha-1)}f(x)\nabla_q x \tag{1.21}$$

It is clear, from the q-analogue of Cauchy’s formula [25], that

$$\nabla_q^n\,_qI_a^n f(t)=f(t) \tag{1.22}$$

where *n* is a positive integer and $0\leq a\in T_q$.

Recently, in Theorem 5 of [15], the authors there have proved that

$$_qI_a^{\beta}\,_qI_a^{\alpha}f(t)=\,_qI_a^{\alpha+\beta}f(t);\alpha,\beta>0. \tag{1.23}$$

The right q-fractional integral of order α is defined by[17]

$$I_q^{\alpha}f(t)=\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)}\int_t^{\infty}(x-t)_q^{(\alpha-1)}f(xq^{1-\alpha})\nabla_q x \tag{1.24}$$

and the right q-fractional integral of order α ending at *b* for some $b\in T_q$ is defined by

$$_bI_q^{\alpha}f(t)=\frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(\alpha)}\int_t^{b}(x-t)_q^{(\alpha-1)}f(xq^{1-\alpha})\nabla_q x \tag{1.25}$$

Please note that, while the left q-fractional integral $_qI_a^{\alpha}$ maps functions defined on T_q to functions defined on T_q, the right q-fractional integral $_bI_q^{\alpha},0<b\leq\infty,$ maps functions defined on $T_q^{1-\alpha}$ to functions defined on T_q.

It is clear, from the q-analogue of Cauchy’s formula [25], that

$$\nabla_q^n I_q^n f(t) = (-1)^n f(t) \tag{1.26}$$

In [24], the author proved that

$$I_q^{\beta} I_q^{\alpha} f(t) = I_q^{\alpha+\beta} f(t); \alpha, \beta > 0. \tag{1.27}$$

Taking into account the domain and the range of the right q-fractional integral, as mentioned above, we note that the above formula is valid under the condition that *f* must be at least defined on T_q, $T_q^{1-\beta}$, $T_q^{1-\alpha}$ and $T_q^{1-(\alpha+\beta)}$.

A particular case of the above identity is

$$I_q^{n-\alpha} I_q^{\alpha} f(t) = I_q^n f(t); \alpha > 0. \tag{1.28}$$

For $\alpha > 0$. If $\alpha \notin N$, then the Caputo left q-fractional and right q-fractional derivatives of order α of a function *f* are, respectively, defined [19] as

$${}_qC_a^{\alpha} f(t) = {}_qI_a^{(n-\alpha)} \nabla_q^n f(t) = \frac{1}{\Gamma_q(n-\alpha)} \int_a^t (t-qx)_q^{n-\alpha-1} \nabla_q^n f(x) \nabla_q x \tag{1.29}$$

and

$${}_bC_q^{\alpha} f(t) = {}_bI_a^{(n-\alpha)} \nabla_q^n f(t) = \frac{q^{-\alpha(\alpha-1)/2}}{\Gamma_q(n-\alpha)} \int_t^b (x-t)_q^{n-\alpha-1} \nabla_q^n f(xq^{1-\alpha}) \nabla_q x \tag{1.30}$$

where $n = [\alpha]+1$. If $\alpha \in N$ then ${}_qC_a^{\alpha} f(t) = \nabla_q^n f(t)$ and ${}_bC_q^{\alpha} f(t) = {}_b\nabla_q^n = (-1)^n \nabla_q^n$.

Also, it is clear that map ${}_qC_a^{\alpha}$ maps functions defined on T_q to functions defined on T_q and that ${}_bC_q^{\alpha}$ maps functions defined on $T_q^{1-\alpha}$ to functions defined on T_q.

Lemma 2 [19]: For any $\alpha > 0$, then

$${}_qI_a^{\alpha} \nabla_q f(t) = \nabla_q {}_qI_a^{\alpha} f(t) - \frac{(t-a)_q^{\alpha-1}}{\Gamma_q(\alpha)} f(a). \tag{1.31}$$

Lemma 3 [19]: For any $0 < \alpha < 1$, then

$${}_qC_a^{\alpha} f(t) = {}_q\nabla_a^{\alpha} f(t) - \frac{(t-a)_q^{-\alpha}}{\Gamma_q(1-\alpha)} f(a). \tag{1.32}$$

Lemma 4[19]: Let $\alpha > 0$ and *f* is defined in suitable domains, then

$${}_qI_a^{\alpha} {}_qC_a^{\alpha} f(t) = f(t) - \sum_{k=0}^{n-1} \frac{(t-a)_q^k}{\Gamma_q(k+1)} \nabla_q^k f(a) \tag{1.33}$$

and if $0 < \alpha \le 1$ then

$${}_qI_a^{\alpha} {}_qC_a^{\alpha} f(t) = f(t) - f(a). \tag{1.34}$$

The following identity is essential to solve linear q-fractional difference equations

$$ {}_qI_a^\alpha (x-a)_q^\mu = \frac{\Gamma_q(\mu+1)}{\Gamma_q(\alpha+\mu+1)}(x-a)_q^{\mu+\alpha}, 0<a<x<b \tag{1.35}$$

where $\alpha \in R^+$ and $\mu \in (-1,\infty)$.

2. q-FRACTIONAL INITIAL VALUE PROBLEM AND q-ANALOGUE OF GENERALIZED MITTAG-LEFFLER FUNCTION:

In this section we will apply the successive approximation method for investigating the solution of the left Caputo q-fractional difference equation in terms of the q-analogue of generalized Mittag-Leffler function.

Example 2.1 Consider the following left Caputo q-fractional difference equation

$$ {}_qC_a^\alpha y(t) = \lambda \frac{(\gamma;q)_1}{[1]_q!} y(t) + f(t), y(a)=\rho, t\in T_q, \tag{2.1}$$

for $0<\alpha\le 1$.

Solution:

Taking ${}_qI_a^\alpha$ defined by (1.21) on the both sides of equation (2.1) and using the formula(1.34), we arrive at

$$ y(t) = \rho + \lambda \frac{(\gamma;q)_1}{[1]_q!} {}_qI_a^\alpha y(t) + {}_qI_a^\alpha f(t). \tag{2.2}$$

To investigate an explicit solution of (2.1), we will apply the following method known as the successive approximation method.

We assume $y_0(t)=\rho$

and

$$ y_m(t) = \rho + \lambda \frac{(\gamma;q)_1}{[1]_q!} {}_qI_a^\alpha y_{m-1}(t) + {}_qI_a^\alpha f(t), m=1,2,3,... \tag{2.3}$$

For $m=1$ by the formula (1.35), we get

$$ y_1(t) = \rho\left(1+\lambda\frac{(\gamma;q)_1 (t-a)_q^\alpha}{[1]_q!\Gamma_q(\alpha+1)}\right) + {}_qI_a^\alpha f(t) \tag{2.4}$$

For $m=2$ we also get

$$ y_2(t) = \rho\left(1+\lambda\frac{(\gamma;q)_1 (t-a)_q^\alpha}{[1]_q!\Gamma_q(\alpha+1)} + \lambda^2\frac{(\gamma;q)_2 (t-a)_q^{2\alpha}}{[2]_q!\Gamma_q(2\alpha+1)}\right) + {}_qI_a^\alpha f(t) + \lambda\frac{(\gamma;q)_1}{[1]_q!} {}_qI_a^{2\alpha} f(t)$$

For $m\to\infty$ we obtain the solution

$$y(t)=\rho\left(1+\sum_{k=1}^{\infty}\frac{\lambda^{k}(\gamma;q)_{k}(t-a)_{q}^{\alpha k}}{[k]_{q}!\Gamma_{q}(\alpha k+1)}\right)+\int_{a}^{t}(t-qx)_{q}^{\alpha-1}\left(\sum_{k=0}^{\infty}\frac{\lambda^{k}(\gamma;q)_{k}(t-q^{\alpha}x)_{q}^{\alpha k}}{[k]_{q}!\Gamma_{q}(\alpha k+\alpha)}\right)f(x)\nabla_{q}x \qquad (2.5)$$

If we compare with the classical case, then the above example suggests the following q-analogue of generalized Mittag-Leffler function defined by Prabhakar [23].

Definition: For $x, x_0, \alpha, \beta, \gamma, \lambda \in C$ and $\mathrm{Re}(\alpha), \mathrm{Re}(\beta), \mathrm{Re}(\gamma) > 0$, the q-analogue of generalized Mittag-Leffler function[23] is defined by

$$E_{\alpha,\beta}^{\gamma}(\lambda, x-x_0; q)=\sum_{k=0}^{\infty}\lambda^{k}\frac{(\gamma;q)_{k}(x-x_0)_{q}^{\alpha k}}{[k]_{q}!\Gamma_{q}(\alpha k+\beta)}. \qquad (2.6)$$

where $[k]_q! = [1]_q[2]_q...[k]_q$ and $[r]_q=\dfrac{1-q^{r}}{1-q}$.

Relationship with another special functions:

(i) If we set $\gamma=1$, we get

$$E_{\alpha,\beta}^{1}(\lambda, x-x_0; q)={}_{q}E_{\alpha,\beta}(\lambda, x-x_0)$$

where ${}_{q}E_{\alpha,\beta}(\lambda, x-x_0)$ is the q-Mittag-Leffler function introduced by Abdeljawad et al.[19].

(ii) If we take $\beta=\gamma=1$, we arrive at

$$E_{\alpha,1}^{1}(\lambda, x-x_0; q)={}_{q}E_{\alpha,1}(\lambda, x-x_0)={}_{q}E_{\alpha}(\lambda, x-x_0)$$

By using the definition(2.6), the equation(2.5) is expressed as

$$y(t)=\rho E_{\alpha,1}^{\gamma}(\lambda, t-a; q)+\int_{a}^{t}(t-qx)_{q}^{\alpha-1}E_{\alpha,\alpha}^{\gamma}(\lambda, t-q^{\alpha}x; q)f(x)\nabla_{q}x. \qquad (2.7)$$

3. SPECIAL CASES:

(i) If we take $\gamma=1$ in above example(2.1), we get the result given by Abdeljawad et al.[19].

(ii)Note that the above proposed definition(2.6) of the q-analogue of generalized Mittag-Leffler function agrees with time scale definition of exponential functions. As it depends on the parameters other than α, β and γ.

(iii)The power term of the q-analogue of generalized Mittag-Leffler function contains α (the term $(x-x_0)_{q}^{\alpha k}$). We include this α in order to express the solution of q-Caputo initial value problem explicitly by means of the q-analogue of generalized Mittag-Leffler function. This is due to that in general it is not true for the q-factorial function to satisfy the power formula $(x-x_0)_{q}^{\alpha k}=[(x-x_0)_{q}^{\alpha}]^{k}$. But for example the latter power formula is true when $x_0=0$. Therefore, for the case $x_0=0$, we may drop α from the power so that the q-analogue of generalized Mittag-Leffler function will tend to the generalized Mittag-Leffler function[23] when $q\to 1$.

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