ORIGINAL PAPER

SOME NEW IDENTITIES IN COMBINATORIC

DAM VAN NHI[1], TRAN TRUNG TINH[1]

Manuscript received15.07.2013; Accepted paper: 28.07.2013;
Published online: 15,09.2013.

Abstract. *In this paper we introduce some new identities in combinatoric.*
Keywords: *Equation, Identity, Combinatoric.*
2010 Mathematics Subject Classification: *26D05, 26D15, 51M16.*

1. INTRODUCTION

Proposition 1.1. Denote $\varphi(x)=\prod_{i=1}^{n}(x+\alpha_i)$. Then, there are the following identities:

(i) $$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\varphi(i)=(-1)^n n!.$$

(ii) $$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^n=(-1)^n n!.$$

(iii) $$\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n+1}{i}=(-1)^n.$$

(iv) $$\sum_{i=1}^{n}\frac{(-1)^i\binom{n}{i}\varphi(i)}{2i-1}=\frac{4n!}{c}+(-1)^n\varphi(0).$$

(v) $$\sum_{i=1}^{n}\frac{(-1)^{i-1}}{2i-1}\binom{n}{i}i^n=\frac{2^n(n!)^2}{(2n)!}.$$

Proposition 1.2. There is the following identity:

$$2\sum_{k=0}^{n}\frac{(-1)^{n-k}k^2\prod_{s=1}^{n}(s^2+k^2)\binom{2n}{n-k}}{1+k^2}=(2n)!.$$

[1] Ha Noi National University of Education, 136 Xuan Thuy Road, Cau Giay, Hanoi, Vietnam.
E-mail: damvannhi@yahoo.com.

Proposition 1.3. For all integer $n \geq 1$, there is the following identity:

$$2\sum_{k=1}^{n}\frac{(-1)^{n-k}\prod_{s=1}^{n}\left(s^2+k^2\right)\binom{2n}{n-k}}{1+k^2}=(2n)!.$$

Proposition 1.4. For all integer $n \geq 1$, there is the following identity:

$$\sum_{k=0}^{n}\left[\frac{\prod_{s=1}^{n}\left(k^2+s^2\right)}{(n!)^2}-1\right]\frac{(-1)^{k-1}2k^2\binom{2n}{n+k}}{1+k^2}=\frac{(2n)!}{\prod_{k=0}^{n}\left(1+k^2\right)}.$$

Proposition 1.5. For all integer $n \geq 1$, we have the following identity:

$$2\sum_{k=0}^{n}\left[(n!)^2-\prod_{r=1}^{n}\left(k^2+r^2\right)\right]\frac{(-1)^{k}\binom{2n}{n-k}}{1+k^2}=\frac{(n!)^4\binom{2n}{n}}{\prod_{k=0}^{n}\left(1+k^2\right)}.$$

2. PROVING SOME NEW IDENTITIES IN COMBINATORIC BY USING THE SYSTEMS OF LINEAR EQUATIONS

Example 2.1. Assume that $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathbb{R}$ and $\alpha_i + j \neq 0,\ i, j = 1, 2, \ldots, n$. Solve the following system of linear equations:

$$\begin{cases} \dfrac{x_1}{1+\alpha_1}+\dfrac{x_2}{2+\alpha_1}+\ldots+\dfrac{x_n}{n+\alpha_1}=1 \\ \dfrac{x_1}{1+\alpha_2}+\dfrac{x_2}{2+\alpha_2}+\ldots+\dfrac{x_n}{n+\alpha_2}=1 \\ \ldots \\ \dfrac{x_1}{1+\alpha_n}+\dfrac{x_2}{2+\alpha_n}+\ldots+\dfrac{x_n}{n+\alpha_n}=1 \end{cases}$$

Proof: Consider $f(x)=\dfrac{x_1}{1+x}+\dfrac{x_2}{2+x}+\ldots+\dfrac{x_n}{n+x}-1=\dfrac{p(x)}{\prod_{i=1}^{n}(i+x)}$, where $p(x)$ is a polynomial of degree *n*. Since $f(\alpha_i)=0$, therefore $p(\alpha_i)=0$, $i=1, \ldots, n$. In view of this result, we get $f(x)=-\dfrac{(x-\alpha_1)(x-\alpha_2)\ldots(x-\alpha_n)}{(x+1)(x+2)\ldots(x+n)}$.

From $\frac{x_1}{1+x}+\frac{x_2}{2+x}+...+\frac{x_n}{n+x}-1=-\frac{(x-\alpha_1)...(x-\alpha_n)}{(x+1)...(x+n)}$ we deduce

$$\begin{aligned}&x_1(x+2)(x+3)...(x+n)+x_2(x+1)(x+3)...(x+n)\\&+x_3(x+1)(x+2)(x+4)...(x+n)+x_4(x+1)...(x+n)\\&+...\\&+x_n(x+1)(x+2)...(x+n-1)-(x+1)(x+2)...(x+n)\\&=-(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)\end{aligned}$$

From straightforward computation, we obtain the solution of the above system equations

$$\begin{cases} x_1=\dfrac{(-1)^{n-1}\prod_{i=1}^{n}(1+\alpha_i)}{(n-1)!} \text{ with } x=-1\\ x_2=\dfrac{(-1)^{n-2}\prod_{i=1}^{n}(2+\alpha_i)}{1!(n-2)!} \text{ with } x=-2\\ x_3=\dfrac{(-1)^{n-3}\prod_{i=1}^{n}(3+\alpha_i)}{2!(n-3)!} \text{ with } x=-3\\ ...\\ x_n=\dfrac{(-1)^{n-n}\prod_{i=1}^{n}(n+\alpha_i)}{(n-1)!} \text{ with } x=-n.\end{cases}$$

Proposition 2.2. Set $\varphi(x)=\prod_{i=1}^{n}(x+\alpha_i)$. Then, there are the following identities:

(i) $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\varphi(i)=(-1)^n n!$.

(ii) $\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^n=(-1)^n n!$.

(iii) $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n+i}{i}=(-1)^n$.

Proof: (i) Similar to Example 2.1, we deduce that: $\sum_{i=1}^{n}\frac{(-1)^{n-i}\varphi(i)}{(x+i)(i-1)!(n-i)!}-1$ $=-\frac{(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)}{(x+1)(x+2)...(x+n)}$. Substitute $x=0$, we have $\sum_{i=1}^{n}\frac{(-1)^{n-i}\varphi(i)}{i!(n-i)!}-1=-\frac{(-1)^{n+1}\varphi(0)}{n!}$ or $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\varphi(i)=(-1)^n n!$

(ii) Substitute $\alpha_1=\ ...\ =\alpha_n=0$ in $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\varphi(i)=(-1)^n n!$ we get

$$\sum_{i=0}^{n}(-1)^i\binom{n}{i}i^n=(-1)^n n!.$$

(iii) Substitute $\alpha_i=i,\ \ i=0,\ 1,\ \ldots,\ n$ we obtain $\sum_{i=0}^{n}(-1)^i\binom{n}{i}\binom{n+i}{i}=(-1)^n.$ □

Example 2.3. Solve the following system of linear equations:

$$\begin{cases}\sum_{k=1}^{n}\frac{sx_k+y_k}{s^2+k^2}=\frac{1}{s}\\ s=\pm 1,\ \ldots,\ \pm n;\quad x_k,\ y_k\in\mathbb{R}\end{cases}$$

Then, evaluate the sum $S=\sum_{k=1}^{n}\frac{\prod_{s=1}^{n}(s^2+k^2)}{\prod_{s\neq k}(k^2-s^2)(1+k^2)}$.

Proof: Consider $f(x)=-\frac{1}{x}+\sum_{k=1}^{n}\frac{x_kx+y_k}{x^2+k^2}=\frac{p(x)}{x\prod_{k=1}^{n}(x^2+k^2)}$, where $p(x)$ is a polynomial of the degree $\leq 2n$. Since $f(s)=0$, we have $p(s)=0$ where $s=\pm 1,\ ...\ ,\ \pm n$. We obtain $f(x)=\frac{a(x^2-1^2)(x^2-2^2)...(x^2-n^2)}{x\prod_{k=1}^{n}(x^2+k^2)}$.

Since $-\frac{1}{x}+\sum_{k=1}^{n}\frac{x_kx+y_k}{x^2+k^2}=\frac{a(x^2-1^2)(x^2-2^2)...(x^2-n^2)}{x\prod_{k=1}^{n}(x^2+k^2)}$ we deduce

$$\begin{aligned}&-(x^2+1^2)...(x^2+n^2)+x(x_1x+y_1)(x^2+2^2)(x^2+3^2)...(x^2+n^2)\\&+x(x_2x+y_2)(x^2+1^2)(x^2+3^2)...(x^2+n^2)+...+x(x_nx+y_n)(x^2+1^2)(x^2+2^2)...(x^2+(n-1)^2)\\&=a(x^2-1^2)(x^2-2^2)...(x^2-n^2)\end{aligned}$$

and obtain the solution
$$\begin{cases} a = -(-1)n \text{ with } x = 0 \\ -x_1 + iy_1 = \dfrac{a(-1)^n \prod_{k=1}^{n}(1^2 + k^2)}{\prod_{k=2}^{n}(k^2 - 1)} \text{ with } x = i \\ -x_2 + iy_2 = \dfrac{a(-1)^n \prod_{k=1}^{n}(2^2 + k^2)}{\prod_{k=2}^{n}(k^2 - 2^2)} \text{ with } x = 2i \\ \dots \\ -x_n + iy_n = \dfrac{a(-1)^n \prod_{k=1}^{n}(n^2 + k^2)}{\prod_{k=2}^{n}(k^2 - n^2)} \text{ with } x = ni \end{cases}$$

Hence $y_1 = \dots = y_n = 0$ and $x_r = \dfrac{\prod_{k=1}^{n}(r^2 + k^2)}{\prod_{k=1,k\neq r}^{n}(k^2 - r^2)}$ with $r = 1, \dots, n$.

Since $-\dfrac{1}{\mathrm{x}} + \sum_{k=1}^{n}\dfrac{x_k x}{x^2 + k^2} = \dfrac{a(x^2 - 1^2)(x^2 - 2^2)\dots(x^2 - n^2)}{x\prod_{k=1}^{n}(x^2 + k^2)}$, we have $\sum_{k=1}^{n}\dfrac{x_k}{1 + k^2} = 1$ or $\sum_{k=1}^{n}\dfrac{\prod_{s=1}^{n}(s^2 + k^2)}{\prod_{s\neq k}^{n}(k^2 - s^2)(1 + k^2)} = 1$ by replacing $x = 1$. □

Proposition 2.4. There is the following identity

$$2\sum_{k=0}^{n}\frac{(-1)^{n-k} k^2 \prod_{s=1}^{n}(s^2 + k^2)\binom{2n}{n-k}}{1 + k^2} = (2n)!.$$

Proof: Since $\sum_{k=1}^{n}\dfrac{\prod_{s=1}^{n}(s^2 + k^2)}{\prod_{s\neq nk}^{n}(k^2 - s^2)(1 + k^2)} = 1$ by Example 2.3. and $\prod_{s\neq nk}^{n}(k^2 - s^2) = \prod_{s\neq nk}^{n}(k - s)$ $\prod_{s\neq nk}^{n}(k + s) = (-1)^{n-k}\dfrac{(n-k)!(n+k)!}{2k^2}$, we have $\sum_{k=1}^{n}\dfrac{2k^2\prod_{s=1}^{n}(s^2 + k^2)}{(-1)^{n-k}(n-k)!(n+k)!(1 + k^2)} = 1$.

By multiplying $(2n)!$ we obtain $2\sum_{k=1}^{n}\frac{(-1)^{n-k}k^2\prod_{s=1}^{n}(s^2+k^2)\binom{2n}{n-k}}{1+k^2}=(2n)!$. □

Example 2.5. Solve the following system of linear equations:

$$\begin{cases}\sum_{k=1}^{n}\frac{x_k}{s-k}+\sum_{k=1}^{n}\frac{y_k}{s+k}=\frac{1}{s}\\ s=\pm i,\ \ldots,\ \pm ni;\ x_k,\ y_k\in\mathbb{R}\end{cases}$$

Then, evaluate the sum $S=\sum_{k=1}^{n}\frac{\prod_{s=1}^{n}(s^2+k^2)}{k^2(1+k^2)\prod_{s\neq k}(k^2-s^2)}$.

Proof: Consider $f(x)=-\frac{1}{x}+\sum_{k=1}^{n}\frac{x_k}{x-k}+\sum_{k=1}^{n}\frac{y_k}{x+k}=\frac{p(x)}{x\prod_{k=1}^{n}(x^2-k^2)}$, where $p(x)$ is a polynomial of the degree $\leq 2n$. Since $f(s)=0$, we have $p(s)=0$ where $s=\pm i,\ \ldots,\ \pm ni$. It is easy to show that $f(x)=\frac{a(x^2+1^2)(x^2+2^2)\ldots(x^2+n^2)}{x\prod_{k=1}^{n}(x^2-k^2)}$.

Since $-\frac{1}{x}+\sum_{k=1}^{n}\frac{x_k}{x-k}+\sum_{k=1}^{n}\frac{y_k}{x+k}=\frac{a(x^2+1^2)(x^2+2^2)\ldots(x^2+n^2)}{x\prod_{k=1}^{n}(x^2-k^2)}$ there is:

$$\begin{aligned}&-(x^2-1^2)\ldots(x^2-n^2)+x_1x(x+1)(x^2-2^2)(x^2-3^2)\ldots(x^2-n^2)+x_2x(x+2)(x^2-1^2)(x^2-3^2)\ldots(x^2-n^2)\\&+\ldots+x_nx(x+n)(x^2-1^2)(x^2-2^2)\ldots(x^2-(n-1)^2)+y_1x(x-1)(x^2-2^2)(x^2-3^2)\ldots(x^2-n^2)\\&+y_2x(x-2)(x^2-1^2)(x^2-3^2)\ldots(x^2-n^2)+\ldots+y_nx(x-n)(x^2-1^2)(x^2-2^2)\ldots(x^2-(n-1)^2)\\&=\ a(x^2+1^2)(x^2+2^2)\ldots(x^2+n^2).\end{aligned}$$

Put $\varphi(x)=\prod_{k=1}^{n}(x^2+k^2)$ and $\psi(x)=\prod_{k=1}^{n}(x^2-k^2)$. Then, we obtain:

$$\begin{cases} a = -(-1)^n \text{ with } x = 0 \\ x_1 = \dfrac{a\prod_{k=1}^{n}(1^2+k^2)}{2\cdot 1^2\prod_{k=2}^{n}(1-k^2)} \text{ with } x = 1 \\ x_2 = \dfrac{a\prod_{k=1}^{n}(2^2+k^2)}{2\cdot 2^2\prod_{k=1,k\neq 2}^{n}(2^2-k^2)} \text{ with } x = 2 \\ \ldots \\ x_n = \dfrac{a\prod_{k=1}^{n}(n^2+k^2)}{2\cdot n^2\prod_{k=1}^{n-1}(n^2-k^2)} \text{ with } x = n \\ y_1 = \dfrac{a\prod_{k=1}^{n}(1^2+k^2)}{2\cdot 1^2\prod_{k=2}^{n}(1-k^2)} \text{ with } x = -1 \\ y_2 = \dfrac{a\prod_{k=1}^{n}(2^2+k^2)}{2\cdot 2^2\prod_{k=1,k\neq 2}^{n}(2^2-k^2)} \text{ with } x = -2 \\ \ldots \\ y_n = \dfrac{a\prod_{k=1}^{n}(n^2+k^2)}{2\cdot n^2\prod_{k=1}^{n-1}(n^2-k^2)} \text{ with } x = -n. \end{cases}$$

From $-\dfrac{1}{x}+\sum_{k=1}^{n}\dfrac{x_k}{x-k}+\sum_{k=1}^{n}\dfrac{y_k}{x+k}=\dfrac{a(x^2+1^2)(x^2+2^2)\ldots(x^2+n^2)}{x\prod_{k=1}^{n}(x^2-k^2)}$ and $x_k = y_k$ it follows $-1+2x^2\sum_{k=1}^{n}\dfrac{x_k}{x^2-k^2}=\dfrac{a(x^2+1^2)(x^2+2^2)\ldots(x^2+n^2)}{\prod_{k=1}^{n}(x^2-k^2)}$.

By replacing $x = i$ we obtain $\sum_{k=1}^{n}\dfrac{\prod_{s=1}^{n}(s^2+k^2)}{k^2(1+k^2)\prod_{s\neq k}(k^2-s^2)}=1$. □

Proposition 2.6. There is the following identity

$$2\sum_{k=1}^{n}\frac{(-1)^{n-k}\prod_{s=1}^{n}\left(s^2+k^2\right)\binom{2n}{n-k}}{1+k^2}=(2n)!.$$

Proof: Since $\sum_{k=1}^{n}\frac{\prod_{s=1}^{n}\left(s^2+k^2\right)}{k^2\left(1+k^2\right)\prod_{s\neq k}\left(k^2-s^2\right)}=1$, by the Example 2.5 and $\prod_{s\neq k}^{n}\left(k^2-s^2\right)=\prod_{s\neq k}^{n}(k-s)\prod_{s\neq k}^{n}(k+s)=(-1)^{n-k}\frac{(n-k)!(n+k)!}{2k^2}$ we have

$$\sum_{k=1}^{n}\frac{2\prod_{s=1}^{n}\left(s^2+k^2\right)}{(-1)^{n-k}(n-k)!(n+k)!\left(1+k^2\right)}=1.$$

We obtain $2\sum_{k=1}^{n}\frac{(-1)^{n-k}\prod_{s=1}^{n}\left(s^2+k^2\right)\binom{2n}{n-k}}{1+k^2}=(2n)!$ by multiplying with $(2n)!$. □

Example 2.7. Solve the following system of linear equations and evaluate the following sum:

(i) $$\begin{cases}\sum_{k=1}^{n}\frac{sx_k+y_k}{s^2+k^2}=\frac{1}{s\left(s^2+1^2\right)\left(s^2+2^2\right)...\left(s^2+n^2\right)}\\ s=\pm 1\ ...\ \pm n;\ x_k,\ y_k\in\mathbb{R}\end{cases}$$

(ii) $$T=\sum_{k=1}^{n}\frac{x_k}{n^2+k^2}.$$

Proof: Consider $f(x)=\sum_{k=1}^{n}\frac{x_kx+y_k}{x^2+k^2}-\frac{1}{x\prod_{k=1}^{n}\left(x^2+k^2\right)}=\frac{p(x)}{x\prod_{k=1}^{n}\left(x^2+k^2\right)}$ with polynomial $p(x)$ of degree $\leq 2n$. Because $f(s)=0$ therefore $p(s)=0$ by replacing $s=\pm 1\ ...\ \pm n$. Hence $f(x)=\frac{a\left(x^2-1^2\right)\left(x^2-2^2\right)...\left(x^2-n^2\right)}{x\prod_{k=1}^{n}\left(x^2+k^2\right)}$.

Since $\sum_{k=1}^{n}\frac{x_kx+y_k}{x^2+k^2}-\frac{1}{x\prod_{k=1}^{n}\left(x^2+k^2\right)}=\frac{a\left(x^2-1^2\right)\left(x^2-2^2\right)...\left(x^2-n^2\right)}{x\prod_{k=1}^{n}\left(x^2+k^2\right)}$ we obtain

$$-1+x(x_1x+y_1)(x^2+2^2)(x^2+3^2)...(x^2+n^2)+x\left(x_2x+y_2\right)(x^2+1^2)(x^2+3^2)...(x^2+n^2)+...$$
$$+\left(x_nx+y_n\right)(x^2+1^2)(x^2+2^2)...(x^2+(n-1)^2)=a(x^2-1^2)(x^2-2^2)...(x^2-n^2)$$

and

$$\begin{cases} a=\dfrac{(-1)^{n-1}}{(n!)^2} \text{ with } x=0 \\ -x_1+iy_1=\dfrac{a(-1)^n\prod\limits_{k=1}^{n}\left(1^2+k^2\right)}{\prod\limits_{k=2}^{n}\left(k^2-1^2\right)}+\dfrac{1}{\prod\limits_{k=2}^{n}\left(k^2-1^2\right)} \text{ with } x=i \\ -x_2+iy_2=\dfrac{a(-1)^n\prod\limits_{k=1}^{n}\left(2^2+k^2\right)}{\prod\limits_{k=1,k\neq2}^{n}\left(k^2-2^2\right)}+\dfrac{1}{\prod\limits_{k=1,k\neq2}^{n}\left(k^2-2^2\right)} \text{ with } x=2i \\ ... \\ -x_n+iy_n=\dfrac{a(-1)^n\prod\limits_{k=1}^{n}\left(n^2+k^2\right)}{\prod\limits_{k=2}^{n-1}\left(k^2-n^2\right)}+\dfrac{1}{\prod\limits_{k=2}^{n}\left(k^2-n^2\right)} \text{ with } x=ni \end{cases}$$

We have $y_k=0$, $x_k=\dfrac{\prod\limits_{s=1}^{n}\left(k^2+s^2\right)}{(n!)^2\prod\limits_{s=1,s\neq k}^{n}\left(s^2-k^2\right)}-\dfrac{1}{\prod\limits_{s=1,s\neq k}^{n}\left(s^2-k^2\right)}$ with $k=1,\ ...\ ,n$ and obtain the identity:

$$\sum_{k=1}^{n}\frac{x_kx}{x^2+k^2}-\frac{1}{x\prod\limits_{k=1}^{n}\left(x^2+k^2\right)}=\frac{(-1)^{n-1}\left(x^2-1^2\right)...\left(x^2-n^2\right)}{(n!)^2x\prod\limits_{k=1}^{n}\left(x^2+k^2\right)}$$

By $x=n$ there is $\sum\limits_{k=1}^{n}\dfrac{x_k}{n^2+k^2}=\dfrac{1}{n^2\prod\limits_{k=1}^{n}\left(n^2+k^2\right)}$. □

Proposition 2.8. There is the following identity:

$$\sum_{k=0}^{n}\left[\frac{\prod\limits_{s=1}^{n}\left(k^2+s^2\right)}{(n!)^2}-1\right]\frac{(-1)^{k-1}2k^2\binom{2n}{n+k}}{1+k^2}=\frac{(2n)!}{\prod\limits_{k=0}^{n}\left(1+k^2\right)}.$$

Proof: By Example 2.7 we have $\sum_{k=1}^{n}\frac{x_k}{1+k^2}=\frac{1}{\prod_{k=1}^{n}\left(1+k^2\right)}$ or the identity

$$\sum_{k=1}^{n}\left[\frac{\prod_{s=1}^{n}\left(k^2+s^2\right)}{(n!)^2\prod_{s=1,s\neq k}^{n}\left(s^2-k^2\right)}-\frac{1}{\prod_{s=1,s\neq k}^{n}\left(s^2-k^2\right)}\right]\frac{1}{1+k^2}=\frac{1}{\prod_{k=1}^{n}\left(1+k^2\right)}$$

Since $\prod_{s=1,s\neq k}^{n}\left(s^2-k^2\right)=\frac{(-1)^{k-1}}{2k^2}(n-k)!(n+k)!$ and $T=\frac{1}{\prod_{k=1}^{n}\left(1+k^2\right)}$ we obtain

$$\begin{aligned}T&=\sum_{k=1}^{n}\left[\frac{\prod_{s=1}^{n}\left(k^2+s^2\right)}{(n!)^2\prod_{s=1,s\neq k}^{n}\left(s^2-k^2\right)}-\frac{1}{\prod_{s=1,s\neq k}^{n}\left(s^2-k^2\right)}\right]\frac{1}{1+k^2}\\&=\sum_{k=1}^{n}\left[\frac{\prod_{s=1}^{n}\left(k^2+s^2\right)}{(n!)^2(n-k)!(n+k)!}-\frac{1}{(n-k)!(n+k)!}\right]\frac{(-1)^{k-1}2k^2}{1+k^2}\\&=\sum_{k=1}^{n}\left[\frac{\prod_{s=1}^{n}\left(k^2+s^2\right)}{(n!)^2(2n)!}-\frac{\binom{2n}{n+k}}{(2n)!}\right]\frac{(-1)^{k-1}2k^2}{1+k^2}\end{aligned}$$

Hence $\sum_{k=1}^{n}\left[\frac{\prod_{s=1}^{n}\left(k^2+s^2\right)}{(n!)^2}-1\right]\frac{(-1)^{k-1}2k^2\binom{2n}{n+k}}{1+k^2}=\frac{(2n)!}{\prod_{k=0}^{n}\left(1+k^2\right)}$. □

Example 2.9. Suppose that all numbers $\alpha_1,\ \alpha_2,\ \ldots,\ \alpha_n$ are different and $\alpha_i+j\neq 0$ for all $i,\ j=1,\ 2,\ \ldots,\ n$. Solve the following system of linear equations:

$$\begin{cases}\frac{x_1}{1+\alpha_1}+\frac{x_2}{2+\alpha_1}+\ldots+\frac{x_n}{n+\alpha_1}=\frac{4}{2\alpha_1+1}\\\frac{x_1}{1+\alpha_2}+\frac{x_2}{2+\alpha_2}+\ldots+\frac{x_n}{n+\alpha_2}=\frac{4}{2\alpha_2+1}\\\ldots\\\frac{x_1}{1+\alpha_n}+\frac{x_2}{2+\alpha_n}+\ldots+\frac{x_n}{n+\alpha_n}=\frac{4}{2\alpha_n+1}.\end{cases}$$

Proof: Consider $f\left(x\right)=\frac{x_1}{1+x}+\frac{x_2}{2+x}+...+\frac{x_n}{n+x}-\frac{4}{2x+1}$. We can write $f\left(x\right)$ as $f\left(x\right)=\frac{p\left(x\right)}{\left(2x+1\right)\prod_{i=1}^{n}\left(i+x\right)}$, where $p\left(x\right)$ is a polynomial of degre $\le n$. Since $f\left(\alpha_i\right)=0$, we have $p\left(\alpha_i\right)=0$, so we obtain $p\left(x\right)=c\frac{\left(x-\alpha_1\right)\left(x-\alpha_2\right)...\left(x-\alpha_n\right)}{\left(x+1\right)\left(x+2\right)...\left(x+n\right)}$, $c\in\mathbb{R}$. Hence

$$f\left(x\right)=-\frac{\left(x-\alpha_1\right)\left(x-\alpha_2\right)...\left(x-\alpha_n\right)}{\left(2x+1\right)\left(x+1\right)\left(x+2\right)...\left(x+n\right)}$$

By the equation

$$\frac{x_1}{1+x}+\frac{x_2}{2+x}+...+\frac{x_n}{n+x}-\frac{4}{2x+1}=\frac{c\left(x-\alpha_1\right)\left(x-\alpha_2\right)...\left(x-\alpha_n\right)}{\left(2x+1\right)\left(x+1\right)\left(x+2\right)...\left(x+n\right)}$$

it induces the following identity

$$\begin{aligned}&(1+2x)\big[x_1(x+2)(x+3)...(x+n)+x_2(x+1)(x+3)...(x+n)\\&+x_3(x+1)(x+2)(x+4)...(x+n)+x_4(x+1)(x+2)...(x+n)\\&+...\\&+x_n(x+1)(x+2)...(x+n-1)\big]-4(x+1)(x+2)...(x+n)\\&=c(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n).\end{aligned}$$

Therefore, we obtain the solution of system

$$\begin{cases} x_1=\dfrac{c\left(-1\right)^{n-1}\prod_{i=1}^{n}\left(1+\alpha_i\right)}{1.\left(n-1\right)!} \text{ with } x=-1\\ x_2=\dfrac{c\left(-1\right)^{n-2}\prod_{i=1}^{n}\left(2+\alpha_i\right)}{3.1!\left(n-2\right)!} \text{ with } x=-2\\ x_3=\dfrac{c\left(-1\right)^{n-3}\prod_{i=1}^{n}\left(3+\alpha_i\right)}{5.2!\left(n-3\right)!} \text{ with } x=-3\\ ...\\ x_n=\dfrac{c\left(-1\right)^{n-n}\prod_{i=1}^{n}\left(n+\alpha_i\right)}{\left(2n-1\right).\left(n-1\right)!} \text{ with } x=-n\\ c=\dfrac{-4\prod_{i=1}^{n}\frac{2i-1}{2}}{\left(-1\right)^{n}\prod_{i=1}^{n}\frac{2\alpha_i+1}{2}} \text{ with } x=\dfrac{-1}{2}. \end{cases}$$

□

Proposition 2.10. Denote $\varphi(x)=\prod_{i=1}^{n}(x+\alpha_i)$. Then, there are some following identities:

(i) $$\sum_{i=1}^{n}\frac{(-1)i\binom{n}{i}\varphi(i)}{2i-1}=\frac{4n!}{c}+(-1)^n\varphi(0).$$

(ii) $$\sum_{i=1}^{n}\frac{(-1)^{i-1}}{2i-1}\binom{n}{i}i^n=\frac{2^n(n!)^2}{(2n)!}.$$

Proof: From the system of equations $\frac{x_1}{1+x}+\frac{x_2}{2+x}+...+\frac{x_n}{n+x}-\frac{4}{2x+1}=\frac{c(x-\alpha_1)(x-\alpha_2)...(x-\alpha_n)}{(2x+1)(x+1)(x+2)...(x+n)}$ we obtain (i) by replacing $x_1,\ ...\ ,\ x_n$ and $x=0$. The result (ii) is deduced from (i). □

Example 2.11. Solve the following system of equations with variables $x_k,\ y_k$:

$$\begin{cases}\sum_{k=1}^{n}\frac{x_k}{s-k}+\sum_{k=1}^{n}\frac{y_k}{s+k}=\frac{1}{s(s^2-1^2)(s^2-2^2)...(s^2-n^2)}\\ s=\pm i,\ \dots,\ \pm ni;\ x_k,\ y_k\in\mathbb{R}\end{cases}$$

and evaluate the sum $\sum_{k=1}^{n}\left[\frac{\prod_{r=1}^{n}(k^2+r^2)}{k^2\prod_{r\neq k}^{n}(k^2-r^2)}-\frac{(n!)^2}{k^2\prod_{r\neq k}^{n}(k^2-r^2)}\right]\frac{1}{1+k^2}$.

Proof: We write $f(x)=\sum_{k=1}^{n}\frac{x_k}{x-k}+\sum_{k=1}^{n}\frac{y_k}{x+k}-\frac{1}{x\prod_{k=1}^{n}(x^2-k^2)}=\frac{p(x)}{x\prod_{k=1}^{n}(x^2-k^2)}$ where the polynomial $p(x)\in\mathbb{R}[x]$ is of degree $\leq 2n$. Since $f(s)=0$, we have $p(s)=0$ where $s=\pm i,\ ...\ ,\ \pm ni$. Hence

$$f(x)=\frac{a(x^2+1^2)(x^2+2^2)...(x^2+n^2)}{x\prod_{k=1}^{n}(x^2-k^2)}$$

Since $\sum_{k=1}^{n}\frac{x_k}{x-k}+\sum_{k=1}^{n}\frac{y_k}{x+k}-\frac{1}{x\prod_{k=1}^{n}(x^2-k^2)}=\frac{a\prod_{k=1}^{n}(x^2+k^2)}{x\prod_{k=1}^{n}(x^2-k^2)}$ we deduce

$$-1+x_1x(x+1)(x^2-2^2)(x^2-3^2)...(x^2-n^2)+x_2x(x+2)(x^2-1^2)(x^2-3^2)...(x^2-n^2)+...$$
$$+x_nx(x+n)(x^2-1^2)(x^2-2^2)...(x^2-(n-1)^2)+y_1x(x-1)(x^2-2^2)(x^2-3^2)...(x^2-n^2)$$
$$+y_2x(x-2)(x^2-1^2)(x^2-3^2)...(x^2-n^2)+...+y_nx(x-n)(x^2-1^2)(x^2-2^2)...(x^2-(n-1)^2)$$
$$=a(x^2+1^2)(x^2+2^2)...(x^2+n^2).$$

To find a and x_k, y_k we replace $x=0,\ 1,\ \dots,\ n$ and obtain

$$\begin{cases} a=\dfrac{-1}{(n!)^2} \text{ with } x=0 \\ x_1=\dfrac{a\prod_{k=1}^{n}\left(1^2+k^2\right)}{2.1^2\prod_{k=2}^{n}\left(1-k^2\right)}+\dfrac{1}{2.1^2\prod_{k=2}^{n}\left(1-k^2\right)} \text{ with } x=1 \\ x_2=\dfrac{a\prod_{k=1}^{n}\left(2^2+k^2\right)}{2.2^2\prod_{k=1,k\neq 2}^{n}\left(2^2-k^2\right)}+\dfrac{1}{2.2^2\prod_{k=1,k\neq 2}^{n}\left(2^2-k^2\right)} \text{ with } x=2 \\ ... \\ x_n=\dfrac{a\prod_{k=1}^{n}\left(n^2+k^2\right)}{2.n^2\prod_{k=1}^{n-1}\left(n^2-k^2\right)}+\dfrac{1}{2.n^2\prod_{k=2}^{n}\left(n^2-k^2\right)} \text{ with } x=n \\ y_1=\dfrac{a\prod_{k=1}^{n}\left(1^2+k^2\right)}{2.1^2\prod_{k=2}^{n}\left(1-k^2\right)}+\dfrac{1}{2.1^2\prod_{k=2}^{n}\left(1-k^2\right)} \text{ with } x=-1 \\ y_2=\dfrac{a\prod_{k=1}^{n}\left(2^2+k^2\right)}{2.2^2\prod_{k=1,k\neq 2}^{n}\left(2^2-k^2\right)}+\dfrac{1}{2.2^2\prod_{k=1,k\neq 2}^{n}\left(2^2-k^2\right)} \text{ with } x=-2 \\ ... \\ y_n=\dfrac{a\prod_{k=1}^{n}\left(n^2+k^2\right)}{2.n^2\prod_{k=1}^{n-1}\left(n^2-k^2\right)}+\dfrac{1}{2.n^2\prod_{k=2}^{n}\left(n^2-k^2\right)} \text{ with } x=-n. \end{cases}$$

From $\sum_{k=1}^{n}\frac{x_k}{x-k}+\sum_{k=1}^{n}\frac{y_k}{x+k}-\frac{1}{x\prod_{k=1}^{n}\left(x^2-k^2\right)}=\frac{a\prod_{k=1}^{n}\left(x^2+k^2\right)}{x\prod_{k=1}^{n}\left(x^2-k^2\right)}$ and $x_k=y_k$ it follows $2x^2\sum_{k=1}^{n}\frac{x_k}{x^2-k^2}-\frac{1}{\prod_{k=1}^{n}\left(x^2-k^2\right)}=\frac{a\prod_{k=1}^{n}\left(x^2+k^2\right)}{\prod_{k=1}^{n}\left(x^2-k^2\right)}$. By replacing $x=i$ we obtain

$$2\sum_{k=1}^{n}\frac{x_k}{1+k^2}=\frac{(-1)^n}{\prod_{k=1}^{n}\left(1+k^2\right)} \text{ or } \frac{(-1)^{n-1}(n!)^2}{\prod_{k=1}^{n}\left(1+k^2\right)}=\sum_{k=1}^{n}\left[\frac{\prod_{r=1}^{n}\left(k^2+r^2\right)}{k^2\prod_{r\neq k}^{n}\left(k^2-r^2\right)}-\frac{(n!)^2}{k^2\prod_{r\neq k}^{n}\left(k^2-r^2\right)}\right]\frac{1}{1+k^2}. \qquad \square$$

Proposition 2.12. For all integer $n\geq 1$ we have the following identity:

$$2\sum_{k=0}^{n}\left[(n!)^2-\prod_{r=1}^{n}\left(k^2+r^2\right)\right]\frac{(-1)^k\binom{2n}{n-k}}{1+k^2}=\frac{(n!)^4\binom{2n}{n}}{\prod_{k=0}^{n}\left(1+k^2\right)}.$$

Proof: By Example 2.11 we obtain the identity

$$\frac{(-1)^{n-1}(n!)^2}{\prod_{k=1}^{n}\left(1+k^2\right)}=\sum_{k=1}^{n}\left[\frac{\prod_{r=1}^{n}\left(k^2+r^2\right)}{k^2\prod_{r\neq k}^{n}\left(k^2-r^2\right)}-\frac{(n!)^2}{k^2\prod_{r\neq k}^{n}\left(k^2-r^2\right)}\right]\frac{1}{1+k^2}$$

By the other way,

$$2k^2\prod_{r\neq k}^{n}\left(k^2-r^2\right)=2k^2\prod_{r\neq k}^{n}(k-r)\prod_{r\neq k}^{n}(k+r)=(-1)^{n-k}(n-k)!(n+k)!$$

we obtain $2\sum_{k=1}^{n}\left[(n!)^2-\prod_{r=1}^{n}\left(k^2+r^2\right)\right]\frac{(-1)^k\binom{2n}{n-k}}{1+k^2}=\frac{(n!)^4\binom{2n}{n}}{\prod_{k=1}^{n}\left(1+k^2\right)}$. Since $\prod_{r=1}^{n}\left(0+r^2\right)=(n!)^2$ there is $2\sum_{k=0}^{n}\left[(n!)^2-\prod_{r=1}^{n}\left(k^2+r^2\right)\right]\frac{(-1)^k\binom{2n}{n-k}}{1+k^2}=\frac{(n!)^4\binom{2n}{n}}{\prod_{k=0}^{n}\left(1+k^2\right)}$. $\square$

REFERENCES

[1] Faddev, D., Sominski, I., *Recueil D'Exercices D'Algbre Suprieure*, Editions Mir-Moscow, 1977.
[2] Rivaud, J., *Exercices D'Algbre 1*, Paris Librairie Vuibert, 1964.