ORIGINAL PAPER

IDENTITIES FOR THE FRACTIONAL HANKEL TRANSFORM

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Abstract. *In the present paper the Parseval-Goldstein type theorems, involving the fractional Hankel, fractional Fourier, and fractional Laplace transform are given. The identities proven in this paper can used to evaluate infinite integral of some special functions.*

Keywords: *fractional Hankel transform, fractional Fourier transform, fractional Laplace transform.*

AMS Subject Classification: *46F12, 44.*

1. INTRODUCTION

Victor Namias [5] introduced and developed the theory of fractional Fourier transform and opened the vast field of fractional integral transform to the researchers. Fractional Fourier transform as defined by him is,

$$F^{\alpha}\left[f(x)\right](y)=\frac{e^{i\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)}}{\sqrt{2\pi\sin\alpha}}\int_0^{\infty}e^{-\frac{i}{2}\left(x^2+y^2\right)\cot\alpha+ixy\csc\alpha}f(x)d\quad x \tag{1.1}$$

In 1980, Namias [6] had further studied fractional Hankel transform which is given by

$$H_v^{\alpha}\left[f(x)\right](y)=\frac{e^{i(v+1)\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)}}{\sin\frac{\alpha}{2}}\int_0^{\infty}xe^{-\frac{i}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}+ixy\csc\alpha}J_v\left(\frac{xy}{\sin\frac{\alpha}{2}}\right)f(x)d \tag{1.2}$$

Noting the limitations of the domain for the above definition by Namias, Kerr F. H. [4], further defined it for the wider domain as,

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$$H_v^\alpha\left[f(x)\right](y)=A_{v,\alpha}\int_0^\infty e^{-\frac{i}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}}\sqrt{\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)f(x)d \tag{1.3}$$

where $A_{v,\alpha}=\left|\sin\frac{\alpha}{2}\right|^{-\frac{1}{2}}e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)}$.

Moreover this fractional Hankel transform is the generalization of Hankel transform given by Zemanian in [13],

$$H\left[f(x)\right](y)=\int_0^\infty\sqrt{xy}\,f(x)J_v(xy)dx \tag{1.4}$$

We have studied some properties of fractional Hankel transform given by (1.3), in [9 – 12]. Fractional Laplace transform which is defined by Sharma K.K. [8] in 2010 as a special case of complex linear canonical transform is given by,

$$L^\alpha\left[f(x)\right](y)=\frac{1}{\sqrt{-2\pi\sin\alpha}}\int_0^\infty e^{-\frac{i}{2}\left(y^2-x^2\right)\cot\alpha-xy\csc\alpha}f(x)d \tag{1.5}$$

will be required in our investigation.

In this paper we have studied different identities involving fractional Hankel transform, as proved by Dernek et al in [1] for Hankel transform.

The paper is organized as follows.

In section II, we find some identities for fractional Hankel transform, these identities are used to prove Parseval Goldstein type theorem, where as in section III illustrative example is given. Last section concludes the paper.

2. IDENTITIES FOR FRACTIONAL HANKEL TRANSFORM:-

First we find Goldstein type exchange identity for the fractional Hankel transform.

Theorem 1. If $\mathrm{Re}V>-1$, then

$$\int_0^\infty H_v^\alpha\left[f(x):y\right]g(y)dy=\int_0^\infty H_v^\alpha\left[g(y):x\right]f(x)dx.$$

Proof: By the definition (1.3) of fractional Hankel transform, we have,

$$\int_0^\infty H_v^\alpha\left[f(x):y\right]g(y)dy=A_{v,\alpha}\int_0^\infty g(y)\left\{\int_0^\infty e^{-\frac{1}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}}\sqrt{\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)f(x)dx\right\}dy$$

by changing the order of integration,

$$\int_0^\infty H_v^\alpha\left[f(x):y\right]g(y)dy = A_{v,\alpha}\int_0^\infty f(x)\left\{A_{v,\alpha}\int_0^\infty e^{-\frac{1}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}}\sqrt{\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)g(y)dy\right\}dx =$$

$$= \int_0^\infty H_v^\alpha\left[g(y):x\right]f(x)dx$$

Now to prove the main Parseval-Goldstein type theorem, we required following Lemma.

Lemma 1: The identity,

$$H_v^\alpha\left\{u^{v+\frac{1}{2}}L^{\frac{\alpha}{2}}\left[e^{ux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\} = C_\alpha\left\{y^{v+\frac{1}{2}}e^{\frac{i\left(1-4\cos^2\frac{\alpha}{2}\right)}{2\sin\alpha}}y^2F^{\frac{\alpha}{2}}\left[e^{ixy\csc\frac{\alpha}{2}}f(x):y\right]\right\} \tag{2.1}$$

where $C_\alpha = \dfrac{e^{-i\left(v\pi+\alpha-\frac{\pi}{2}\right)+i\left(\frac{v+1}{2}\right)\left(-\alpha+\pi\hat{\alpha}\right)}}{\sin\frac{\alpha}{2}2^{v+1}\cos^{v+1\frac{\alpha}{2}}}$, hold true provided that $\mathrm{Re}V > -1$ and each member of above equation exists.

Proof: Using the definition (1.3) of the fractional Hankel transform and (1.5) of the fractional Laplace transform, we have

$$H_v^\alpha\left\{u^{v+\frac{1}{2}}L^{\frac{\alpha}{2}}\left[e^{ux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\} =$$

$$= A_{v,\alpha}\int_0^\infty e^{-\frac{i}{2}\left(u^2+y^2\right)\cot\frac{\alpha}{2}}u^{v+\frac{1}{2}}\sqrt{\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}\right)L^{\frac{\alpha}{2}}\left[e^{ux\csc\frac{\alpha}{2}}f(x):u\right]du = \tag{2.2}$$

$$= \frac{A_{v,\alpha}}{\sqrt{-2\pi\sin\frac{\alpha}{2}}}\int_0^\infty e^{-\frac{i}{2}\left(u^2+y^2\right)\cot\frac{\alpha}{2}}u^{v+\frac{1}{2}}\sqrt{\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}\right)\int_0^\infty e^{-\frac{i}{2}\left(u^2-x^2\right)\cot\frac{\alpha}{2}}f(x)dxdu$$

Changing the order of integration in equation (2.2), we have,

$$H_v^\alpha\left\{u^{v+\frac{1}{2}}L^{\frac{\alpha}{2}}\left[e^{ux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\}=$$

$$=\frac{A_{v,\alpha}}{\sqrt{-2\pi\sin\frac{\alpha}{2}}}\int_0^\infty e^{-\frac{i}{2}\left(y^2-x^2\right)\cot\frac{\alpha}{2}}f(x)\left[\int_0^\infty e^{-iu^2\cot\frac{\alpha}{2}}u^{v+1}\sqrt{\frac{y}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}\right)\right]du\ dx$$

using the Result 12, P.No.30 of [2] in above equation we have,

$$H_v^\alpha\left\{u^{v+\frac{1}{2}}L^{\frac{\alpha}{2}}\left[e^{ux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\}=$$

$$=\frac{A_{v,\alpha}}{\sqrt{-2\pi\sin\frac{\alpha}{2}}}\int_0^\infty e^{-\frac{i}{2}\left(y^2-x^2\right)\cot\frac{\alpha}{2}}f(x)\left[\frac{y^{v+\frac{1}{2}}\sec^{v+1}\frac{\alpha}{2}}{\sqrt{\sin\frac{\alpha}{2}}\,2^{v+1}}(-i)e^{-iv\frac{\pi}{2}}e^{\frac{iy^2}{2\sin\alpha}}\right]dx=$$

$$=\frac{-iA_{v,\alpha}}{\sqrt{\sin\frac{\alpha}{2}}}\frac{\sec^{v+1}\frac{\alpha}{2}}{2^{v+1}}e^{-i\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}e^{-iv\frac{\pi}{2}}y^{v+\frac{1}{2}}e^{iy^2\left[\frac{1}{2\sin\alpha}-\cot\frac{\alpha}{2}\right]}\frac{e^{i\left(\frac{\pi}{4}+\frac{\alpha}{2}\right)}}{\sqrt{-2\pi\sin\frac{\alpha}{2}}}\int_0^\infty e^{\frac{i}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}}f(x)dx$$

Using the definition (1.1) of the fractional Fourier transform, we have

$$H_v^\alpha\left\{u^{v+\frac{1}{2}}L^{\frac{\alpha}{2}}\left[e^{ux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\}=C_\alpha\left\{y^{v+\frac{1}{2}}e^{\frac{i\left(1-4\cos^2\frac{\alpha}{2}\right)}{2\sin\alpha}y^2}F^{-\frac{\alpha}{2}}\left[e^{ixy\csc\frac{\alpha}{2}}f(x):y\right]\right\}$$

where $C_\alpha=\dfrac{e^{-i\left(v\pi+\alpha-\frac{\pi}{2}\right)+i\left(\frac{v+1}{2}\right)(\pi\hat{\alpha}-\alpha)}}{\sin\frac{\alpha}{2}2^{v+1}\cos^{v+1}\frac{\alpha}{2}}$.

Theorem 2. The following Parseval-Goldstein type identities hold true,

$$\int_0^\infty H_v^\alpha\left[f(x):y\right]\left\{L^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{uy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy=$$

$$=C_\alpha\int_0^\infty\left\{f(x)x^{v+\frac{1}{2}}e^{i\left(\frac{1}{2}\csc\alpha-\cot\frac{\alpha}{2}\right)x^2}F^{-\frac{\alpha}{2}}\left[e^{ixu\csc\frac{\alpha}{2}}g(u):x\right]\right\}dx$$

where $C_\alpha = \dfrac{e^{-i\left(v\pi+\alpha-\frac{\pi}{2}\right)+i\left(\frac{v+1}{2}\right)(\pi\hat{\alpha}-\alpha)}}{\sin\frac{\alpha}{2}2^{v+1}\cos^{v+1\frac{\alpha}{2}}}$ provided that $\operatorname{Re}V > -1$ and the integrals converge absolutely.

Proof: Using definition (1.3) and (1.5), we have,

$$\int_0^\infty H_v^\alpha\left[f(x):y\right]\left\{L^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{uy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy =$$

$$=\int_0^\infty\left[\int_0^\infty A_{v,\alpha}e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}\sqrt{\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)f(x)dx\right]\left\{L^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{uy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy$$

Changing the order of integration which is permissible under the assumptions of the theorem.

$$\int_0^\infty H_v^\alpha\left[f(x):y\right]\left\{L^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{uy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy =$$

$$=\int_0^\infty f(x)\left[\int_0^\infty A_{v,\alpha}e^{-\frac{i}{2}(x^2+y^2)\cot\frac{\alpha}{2}}\sqrt{\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)\left\{L^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{uy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy\right]=$$

$$=\int_0^\infty f(x)H_v^\alpha\left\{L^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{uy\csc\frac{\alpha}{2}}g(u):y\right]:x\right\}dx$$

using result (2.1) of Lemma 1, we have,

$$\int_0^\infty H_v^\alpha\left[f(x):y\right]\left\{L^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{uy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy =$$

$$=\int_0^\infty f(x)\left\{C_\alpha x^{v+\frac{1}{2}}e^{i\left(\frac{1-4\cos^2\frac{\alpha}{2}}{2\sin\alpha}\right)}F^{-\frac{\alpha}{2}}\left[g(u)e^{ixu\csc\frac{\alpha}{2}}:x\right]\right\}dx =$$

$$=C_\alpha\int_0^\infty\left\{f(x)x^{v+\frac{1}{2}}e^{i\left\{\frac{1}{2}\csc\alpha-\cot\frac{\alpha}{2}\right\}x^2}F^{-\frac{\alpha}{2}}\left[g(u)e^{ixu\csc\frac{\alpha}{2}}:x\right]\right\}dx$$

hence proved.

Lemma 2. The identity

$$H_v^{\alpha}\left\{u^{v+\frac{1}{2}}F^{\frac{\alpha}{2}}\left[e^{-iux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\}=B_{\alpha}\left\{y^{v+\frac{1}{2}}e^{iy^2\left(\frac{1}{4}\tan\frac{\alpha}{2}+\cot\frac{\alpha}{2}\right)}L^{\frac{\alpha}{2}}\left[e^{-xy\csc\frac{\alpha}{2}}f(x):y\right]\right\} \qquad (2.3)$$

where $B_{\alpha}=\left|\sin\frac{\alpha}{2}\right|^{-\frac{1}{2}}e^{i\,(v+1)\left[\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}-\frac{\pi}{2}\right]}e^{i\left(\frac{3\pi}{4}-\frac{\alpha}{4}\right)}$, hold true provided that $\mathrm{Re}V>-1$ and each member of the above assertion exists.

Proof: Using definition (1.1) and (1.3), we have,

$$H_v^{\alpha}\left\{u^{v+\frac{1}{2}}F^{\frac{\alpha}{2}}\left[e^{-iux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\}=$$

$$=A_{v,x}\int_0^{\infty}e^{-\frac{i}{2}\left(u^2+y^2\right)\cot\frac{\alpha}{2}}\sqrt{\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}\right)\left\{F^{\frac{\alpha}{2}}\left[u^{v+\frac{1}{2}}e^{-iux\csc\frac{\alpha}{2}}f(x):u\right]\right\}du=$$

$$=A_{v,x}\int_0^{\infty}e^{-\frac{i}{2}\left(u^2+y^2\right)\cot\frac{\alpha}{2}}\sqrt{\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}\right)\frac{e^{i\left\{\frac{\pi}{4}-\frac{\alpha}{4}\right\}}}{\sqrt{2\pi\sin\frac{\alpha}{2}}}\left\{\int_0^{\infty}e^{-\frac{i}{2}\left(x^2+u^2\right)\cot\frac{\alpha}{2}+iux\csc\frac{\alpha}{2}}u^{v+\frac{1}{2}}e^{-iux\csc\frac{\alpha}{2}}f(x)\right\}dxdu$$

Changing the order of integration,

$$H_v^{\alpha}\left\{u^{v+\frac{1}{2}}F^{\frac{\alpha}{2}}\left[e^{-iux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\}=$$

$$=A_{v,x}\frac{e^{i\left\{\frac{\pi}{4}-\frac{\alpha}{4}\right\}}}{\sqrt{2\pi\sin\frac{\alpha}{2}}}\int_0^{\infty}f(x)e^{-\frac{i}{2}\left(x^2+u^2\right)\cot\frac{\alpha}{2}}\left[\int_0^{\infty}e^{-iu^2\cot\frac{\alpha}{2}}u^{v+\frac{1}{2}}\sqrt{\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}}J_v\left(\frac{uy}{\left|\sin\frac{\alpha}{2}\right|}\right)du\right]dx$$

Using the result (12), page no. 30 of [2], we have

$$H_v^{\alpha}\left\{u^{v+\frac{1}{2}}F^{\frac{\alpha}{2}}\left[e^{-iux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\}=$$

$$=A_{v,x}\frac{e^{i\left\{\frac{\pi}{4}-\frac{\alpha}{4}\right\}}}{\sqrt{2\pi\sin\frac{\alpha}{2}}}\int_0^{\infty}f(x)e^{-\frac{i}{2}\left(x^2+u^2\right)\cot\frac{\alpha}{2}}\left[\frac{y^{v+\frac{1}{2}}}{\left(2\cot\frac{\alpha}{2}\right)^{v+1}}e^{-i\left(\frac{v+1}{2}\pi-\frac{y^2}{4\cot\frac{\alpha}{2}}\right)}\right]dx=$$

$$=\frac{A_{v,x}}{\left(2\cot\frac{\alpha}{2}\right)^{v+1}}e^{i\left\{\frac{\pi}{4}-\frac{\alpha}{4}\right\}}e^{-i(v+1)\frac{\pi}{2}}y^{v+\frac{1}{2}}e^{iy^2\left\{\frac{1}{4\cot\frac{\alpha}{2}}+\cot\frac{\alpha}{2}\right\}}\frac{1}{\sqrt{2\pi\sin\frac{\alpha}{2}}}\int_0^{\infty}e^{\frac{i}{2}\left(y^2-x^2\right)\cot\frac{\alpha}{2}}\left[f(x)e^{-xy\csc\frac{\alpha}{2}}\right]e^{xy\csc\frac{\alpha}{2}}dx$$

using definition (1.5), we have,

$$H_v^{\alpha}\left\{u^{v+\frac{1}{2}}F^{\frac{\alpha}{2}}\left[e^{-iux\csc\frac{\alpha}{2}}f(x):u\right]:y\right\}=B_{\alpha}\left\{y^{v+\frac{1}{2}}e^{iy^2\left(\frac{1}{4}\tan\frac{\alpha}{2}+\cot\frac{\alpha}{2}\right)}L^{\frac{\alpha}{2}}\left[e^{-xy\csc\frac{\alpha}{2}}f(x):y\right]\right\}$$

hence proved .

Theorem 3. The following Parseval-Goldstein type identities hold true,

$$\int_0^{\infty}H_v^{\alpha}\left[f(x):y\right]\left\{F^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{-iuy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy=$$

$$B_{\alpha}\int_0^{\infty}f(x)\left\{x^{v+\frac{1}{2}}e^{ix^2\left(\frac{1}{3}\tan\frac{\alpha}{2}+\cot\frac{\alpha}{2}\right)}L^{\frac{\alpha}{2}}\left[e^{-xu\csc\frac{\alpha}{2}}g(u):x\right]\right\}dx$$

where $B_{\alpha}=\left|\sin\frac{\alpha}{2}\right|^{-\frac{1}{2}}e^{i\,(v+1)\left[\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}-\frac{\pi}{2}\right]}e^{i\left(\frac{3\pi}{4}-\frac{\alpha}{4}\right)}$, hold true provided that $\operatorname{Re}V>-1$ and each member of the above assertion exists.

Proof: Using definition (1.1), (1.3) we have,

$$\int_0^{\infty}H_v^{\alpha}\left[f(x):y\right]\left\{F^{\frac{\alpha}{2}}\left[y^{v+\frac{1}{2}}e^{-iuy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy=$$

$$=A_{v,\alpha}\int_0^{\infty}F^{\frac{\alpha}{2}}\left\{y^{v+\frac{1}{2}}e^{-iuy\csc\frac{\alpha}{2}}g(u):y\right\}\int_0^{\infty}e^{-\frac{i}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}}J_v\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)\sqrt{\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}}f(x)dxdy$$

Changing the order of integration which is permissible under the assumptions of the theorem,

$$\int_0^\infty H_\nu^\alpha\left[f(x):y\right]\left\{F^{\frac{\alpha}{2}}\left[y^{\nu+\frac{1}{2}}e^{-iuy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy=$$

$$=A_{\nu,\alpha}\int_0^\infty f(x)\left[\int_0^\infty\left\{F^{\frac{\alpha}{2}}\left[y^{\nu+\frac{1}{2}}e^{-iuy\csc\frac{\alpha}{2}}g(u):y\right]\right\}e^{-\frac{i}{2}\left(x^2+y^2\right)\cot\frac{\alpha}{2}}\sqrt{\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}}J_\nu\left(\frac{xy}{\left|\sin\frac{\alpha}{2}\right|}\right)dy\right]dx=$$

$$=\int_0^\infty f(x)H_\nu^\alpha\left\{F^{\frac{\alpha}{2}}\left[y^{\nu+\frac{1}{2}}e^{-iuy\csc\frac{\alpha}{2}}g(u):y\right]:x\right\}dx$$

Using result (2.3) of Lemma 2, we have,

$$\int_0^\infty H_\nu^\alpha\left[f(x):y\right]\left\{F^{\frac{\alpha}{2}}\left[y^{\nu+\frac{1}{2}}e^{-iuy\csc\frac{\alpha}{2}}g(u):y\right]\right\}dy=$$

$$=\int_0^\infty f(x)\left\{B_\alpha x^{\nu+\frac{1}{2}}e^{ix^2\left(\frac{1}{4}\tan\frac{\alpha}{2}+\cot\frac{\alpha}{2}\right)}L^{-\frac{\alpha}{2}}\left[e^{-xu\csc\frac{\alpha}{2}}g(u):x\right]\right\}dx=$$

$$=B_\alpha\int_0^\infty f(x)\left\{x^{\nu+\frac{1}{2}}e^{ix^2\left(\frac{1}{4}\tan\frac{\alpha}{2}+\cot\frac{\alpha}{2}\right)}L^{-\frac{\alpha}{2}}\left[e^{-xu\csc\frac{\alpha}{2}}g(u):x\right]\right\}dx$$

hence the proof.

3. ILLUSTRATIVE EXAMPLE

Using results of the last section, we present an interesting illustration for the identity of Lemma 2,

Example:
We show that,

$$L^{\frac{\alpha}{2}}\left[e^{-\frac{x^2}{2}}e^{(1-i)xy\csc\frac{\alpha}{2}}:y\right]=C_\alpha e^{\frac{-y^2}{4}\left(\csc^2\frac{\alpha}{2}-i\tan\frac{\alpha}{2}-2i\cot\frac{\alpha}{2}\right)},$$

where, $C_\alpha=\dfrac{e^{i\left((\nu+1)\frac{\pi}{2}-\frac{3\pi}{4}-\frac{\alpha}{4}\right)}}{\sin^{\nu+1}\left(-\frac{\alpha}{2}\right)}$.

Proof: We put $f(x)=e^{iux\csc\frac{\alpha}{2}}e^{-\frac{x^2}{2}}$ in L.H.S. of Result (2.3) of Lemma 2, we have,

$$H_v^{\alpha}\left\{u^{v+\frac{1}{2}}F^{\frac{\alpha}{2}}\left[e^{-iux\csc\frac{\alpha}{2}}f\left(x\right):u\right]:y\right\}=H_v^{\alpha}\left\{u^{v+\frac{1}{2}}F^{\frac{\alpha}{2}}\left[e^{-\frac{x^2}{2}}:u\right]:y\right\}$$

Using Result (2) of [5], we have,

$$H_v^{\alpha}\left\{u^{v+\frac{1}{2}}F^{\frac{\alpha}{2}}\left[e^{-\frac{x^2}{2}}:u\right]:y\right\}=H_v^{\alpha}\left\{u^{v+\frac{1}{2}}e^{-\frac{u^2}{2}}:y\right\}$$

Using Result (2) of [11], we have

$$H_v^{\alpha}\left\{u^{v+\frac{1}{2}}e^{-\frac{u^2}{2}}:y\right\}=\frac{\left|\sin\frac{\alpha}{2}\right|^{-\frac{1}{2}}}{\left|\sin\frac{\alpha}{2}\right|^{v+\frac{1}{2}}}y^{v+\frac{1}{2}}e^{-y^2\left(\frac{1}{4}\csc^2\frac{\alpha}{2}+\frac{i}{2}\cot\frac{\alpha}{2}\right)}$$

Using Result (2.3) of Lemma 2, we have ,

$$\frac{e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}\right)(v+1)}}{\left|\sin\frac{\alpha}{2}\right|^{v+1}}y^{v+\frac{1}{2}}e^{-y^2\left(\frac{1}{4}\csc^2\frac{\alpha}{2}+\frac{i}{2}\cot\frac{\alpha}{2}\right)}=\left|\sin\frac{\alpha}{2}\right|^{-\frac{1}{2}}e^{i\left(\frac{\pi}{2}\hat{\alpha}-\frac{\alpha}{2}-\frac{\pi}{2}\right)(v+1)}e^{i\left(\frac{3\pi}{4}-\frac{\alpha}{4}\right)}y^{v+\frac{1}{2}}e^{\frac{iy^2}{4}\tan\frac{\alpha}{2}}L^{\frac{\alpha}{2}}\left[e^{-\frac{x^2}{2}}e^{ixy\csc\frac{\alpha}{2}}e^{xy\csc\frac{\alpha}{2}}:y\right]$$

gives us,

$$L^{\frac{\alpha}{2}}\left[e^{-\frac{x^2}{2}}e^{(1-i)xy\csc\frac{\alpha}{2}}:y\right]=\frac{e^{i\left((v+1)\frac{\pi}{2}-\frac{3\pi}{4}-\frac{\alpha}{4}\right)}}{\sin^{v+\frac{1}{2}}\left(-\frac{\alpha}{2}\right)}e^{\frac{-y^2}{4}\left(\csc^2\frac{\alpha}{2}-i\tan\frac{\alpha}{2}-2i\cot\frac{\alpha}{2}\right)}=C_{\alpha}e^{\frac{-y^2}{4}\left(\csc^2\frac{\alpha}{2}-i\tan\frac{\alpha}{2}-2i\cot\frac{\alpha}{2}\right)}$$

CONCLUSIONS

The theory of fractional integral transform has many applications in signal processing [7], quantum mechanics [5], optics [3, 7]. Hence it is important and interesting to study them. Here we have proved some identities for fractional Hankel transform with fractional Fourier transform and fractional Laplace transform. Parseval Goldstein type identities and some Goldstein exchange formulae are developed. Using these identities some integrals are solved.

The method can also be applied to other fractional integral transform and these identities will be useful to solve some infinite integrals.

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