ORIGINAL PAPER

A CLASS OF FUNCTIONAL EQUATIONS FOR INVOLUTIVE AUTOMORPHISMS OF *n* – GROUPS

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Abstract. *In a* n*-group* $(G,[\ ])$ *consider the following functional equations:*

(E_p) : $$f([x_1,\ldots,x_p, f(x_{p+1}),\ldots, f(x_n)]) = [f(x_1),\ldots, f(x_p), x_{p+1},\ldots, x_n]$$

$(x_1,\ldots,x_p,x_{p+1},\ldots,x_n \in G,\ 1 \le p \le n-1)$. *In the case of groups these equations were studied by I. Corovei and V. Pop* [4]. *We study this class of equations and characterize their solutions using involutive automorphisms of n-groups.*

Keywords: *functional equation, automorphisms, n-groups.*

1. INTRODUCTION

On of the most efficient tool in the theory of *n*-groups is the reducing method, in order to use known results from group theory. By Hosszú theorem [3], we associate to a *n*-group a family of reduced groups, all of these giving by extension, the initial *n*-group.

In this paper we will use these methods and we reveal the necessary results and notions.

Let $(G,[\ ])$ be a *n*-group with the *n*-ary operation $[\ ]: G^n \to G$ and let us denote by $\bar{e}$ the skew element of $e \in G$.

For every $e \in G$ we define the binary operation on G by

$$x \cdot y = [x, \underset{n-3}{e}, \bar{e}, y],\ x, y \in G. \tag{2.1}$$

The pair $(G,\cdot)$ is a group, which is called the reduced group in Hosszú sense, and we denote $(G,\cdot) = \mathrm{Red}_e(G,[\ ])$.

M. Hosszú [3] has proved that the function

$$\alpha_e : G \to G,\ \alpha_e(x) = [e, x, \underset{n-3}{e}, \bar{e}],\ x \in G, \tag{2.2}$$

is an automorphism of $(G,\cdot)$, α^{n-1} is an inner automorphism

$$\alpha_e^{n-1}(x) = a \cdot x \cdot a^{-1},\ x \in G, \text{ where } a = [\underset{n}{e}]. \tag{2.3}$$

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Conversely: If $(G,\cdot)$ is a group, for every pair (α, a), where $a \in G$, α is an automorphism of $(G,\cdot)$, $\alpha^{n-1}(x) = a \cdot x \cdot a^{-1}$, $x \in G$, then the *n*-ary operation

$$[x_1,...,x_n] = x_1 \cdot \alpha(x_2) \cdot ... \cdot \alpha^{n-1}(x_n) \cdot a \qquad (2.4)$$

determines on G a *n*-group structure that is called *n*-ary extension of $(G,\cdot)$ in Hosszú sense and is denoted by

$$(G,[\ ]) = \mathrm{Ext}_{\alpha,a}(G,\cdot)\ ..$$

M. Hosszú [3] show that

$$\mathrm{Ext}_{\alpha_e,a}(\mathrm{Red}_e(G,[\ ])) = (G,[\ ]) \qquad (2.5)$$

(every *n*-ary operation which determines a *n*-group is of the form (1.4)).
The relations between the morphisms of *n*-groups and its reduced group was established in [1, 2].

Theorem. [2] A map $f : G \to G$ is a morphism of *n*-group $(G,[\ ]) = \mathrm{Ext}_{\alpha,a}(G,\cdot)$ iff there exist a binary morphism of groups $(G,\cdot)$, $g : G \to G$ such that:

$$\begin{aligned} &\text{a) } f(x) = f(e) \cdot g(x),\ x \in G; \\ &\text{b) } g(\alpha(x)) = \alpha(f(e)) \cdot \alpha(g(x)) \cdot (\alpha(f(e)))^{-1}; \\ &\text{c) } f(a) = [\underset{n}{f(e)}]. \end{aligned} \qquad (2.6)$$

We recall that a function $f : G \to G$ is involutive if $f \circ f = 1_G$ or $f = f^{-1}$. (2.7)

2. THE FUNCTIONAL EQUATIONS $f : G \to G$

(E_p) : $\quad f([x_1,...,x_p, f(x_{p+1}),..., f(x_n)]) = [f(x_1),..., f(x_p), x_{p+1},..., x_n]$

ON A n-GROUP $(G,[\ ])$

Let $(G,[\ ])$ be a *n*-group and $1 \le p \le n-1$. We consider the functional equation on G:

$$(E_p) \qquad \begin{cases} f : G \to G \\ f([x_1,...,x_p, f(x_{p+1}),..., f(x_n)]) = [f(x_1),..., f(x_p), x_{p+1},..., x_n] \end{cases}$$

$x_1,...,x_p, x_{p+1},..., x_n \in G$.

For $e \in G$ we consider the Hosszú reduced group $(G,\cdot) = \mathrm{Red}_e(G,[\ ])$ using the same notation as (1.1), (1.2), (1.4).

Theorem 2.1. If the function $f : G \to G$ verifies the equation (E_p) in the n-group $(G,[\ ])$ then f verifies the equation (2.1) on the group $(G,\cdot)$

$$f(x \cdot b \cdot f(y)) = f(x) \cdot c \cdot y,\ x, y \in G \tag{2.1}$$

where $b = \alpha^p(f(e)) \cdot \alpha^{p+1}(f(e)) \cdot \ldots \cdot \alpha^{n-2}(f(e)) \cdot a$, $c = \alpha(f(e)) \cdot \alpha^2(f(e)) \cdot \ldots \cdot \alpha^{p-1}(f(e)) \cdot a$.

Proof: Taking in (E_p) $x_1 = x$, $x_2 = \ldots = x_{n-1} = e$, $x_n = y$ and using $\alpha(e) = \ldots = \alpha^{n-1}(e) = e$, $\alpha(\bar{e}) = \bar{e}$, $\alpha^{n-1}(x) = a \cdot x \cdot a^{-1}$ we obtain the equation (2.1). □

Theorem 2.2. If the function f verifies the equation (2.1) then f is a bijection and the function $g : G \to G$ defined by $f(x) = g(y_0^{-1} \cdot x)$, $x \in G$, satisfies the equation

$$g(u \cdot g(v)) = g(u) \cdot v,\ u, v \in G, \tag{2.2}$$

where $y_0 \in G$ satisfies $f(y_0) = e$.

Proof: Taking in (2.1) $a = b^{-1}$ it follows $f(f(y)) = f(b^{-1}) \cdot c \cdot y$, $y \in G$ and since the translation $h : G \to G$, $h(y) = f(b^{-1}) \cdot c \cdot y$, $y \in G$ is a bijection it follows that f is bijection too.

Let now $y = y_0$ in (1) where $f(y_0) = e$.

It follows $f(x \cdot b) = f(x) \cdot c \cdot y_0$, $x \in G$.

The equation (2.1) becomes $f(xbf(y)) = f(xb) \cdot y_0^{-1} \cdot y$, $x, y \in G$. By the transformation $f(x) = g(y_0^{-1} \cdot x)$, $x \in G$ we obtain: $g(y_0^{-1} \cdot xb \cdot g(y_0^{-1} \cdot y)) = g(y_0^{-1} \cdot xb) y_0^{-1} \cdot y$, $x, y \in G$. Denoting $u = y_0^{-1} \cdot x \cdot b$, $v = y_0^{-1} \cdot y$ it follows: $g(u \cdot g(v)) = g(u) \cdot v$, for every $u, v \in G$. □

Theorem 2.3. The function $g : G \to G$ satisfies the equation (2.2) iff g is an involutive automorphism of the group $(G,\cdot)$.

Proof: Taking in (2.2) $u = e$ it follows $g(g(v)) = g(e) \cdot v$, $v \in G$, so g is a morphism. For $u = v = e$ it follows $g(g(e)) = g(e)$, therefore $g(e) = e$ and $g(g(v)) = v$, $v \in G$ (g is idempotent). The equation (2.2) becomes: $g(u \cdot g(v)) = g(u) \cdot g(g(v))$, $u, v \in G$ or $g(u \cdot t) = g(u) \cdot g(t)$, for all $u \in G$, $t = g(v) \in G$, so g is a morphism.

Remark. The solutions of the functional equation

$$g : \mathbf{R} \to \mathbf{R},\ g(x + g(y)) = g(x) + y,\ x, y \in \mathbf{R}$$

are the additive functions which on a Hamel basis H are defined as follows:

Let H be partitioned as $H = H_0 \cup H_1 \cup H_2$ and suppose that there exists a bijection $\varphi : H_1 \to H_2$.

We define $f(h_0) = h_0$, $h_0 \in H_1$, $f(h_1) = \varphi(h_1)$, $h_1 \in H_1$, $f(h_2) = \varphi^{-1}(h_2)$, $h_2 \in H_2$.

Theorem 2.4. The function f satisfies the equation (2.1) iff $f(x) = d \cdot g(x)$, $x \in G$, where $d \in G$, $g : G \to G$ is an involutive automorphism of $(G,\cdot)$ and $g(b \cdot d) = c$, b,c are defined in Theorem 2.1.

Proof: From Theorem 2.2 and Theore, 2,3 we have

$$f(x) = g(y_0^{-1} \cdot x) = g(y_0^{-1}) \cdot g(x) = d \cdot g(x),\ x \in G,$$

where $d = g(y_0^{-1}) = f(e)$ and g is an idempotent automorphism. Taking account of (2.1) we obtain: $dg(xbdg(y)) = dg(x)cy$ or $dg(x)g(bd)g(g(y)) = dg(x)cy$ or $g(bd) = c$. □

Theorem 2.5. If the function f satisfies the equation (E_p) on the n-group $(G,[\ ])$ then there exists an element $d \in G$, an automorphism g of bigroup $(G,\cdot) = \mathrm{Red}_e(G,[\ ])$ such that:

a) $f(x) = d \cdot g(x)$, $x \in G$;
b) $g(g(x)) = x$, $x \in G$;
c) $f([\underset{r}{e}, \underset{n-r}{d}]) = [\underset{r}{d}, \underset{n-r}{e}]$.

Proof: Using Theorems 2.1, 2.2 and 2.3 it follows a) and b).

The relation c) is $f([\underset{r}{e}, f(\underset{n-r}{e})]) = [\underset{r}{f(e)}, \underset{n-r}{e}]$ which is the same with $g(b \cdot d) = c$ from Theorem 2.4.

Remark. Taking in (E_p) $x_1 = e$, $x_2 = x$, $x_3 = \ldots = x_n = e$ we can prove the relation:

$$g(\alpha(x)) = \alpha(f(e)) \cdot \alpha(g(x)) \cdot (\alpha(f(e)))^{-1},\ x \in G,$$

which is a necessary condition (Theorem 1.6), that f to be morphism of n-group $(G,[\ ])$, but it is not sufficiently (the relation c) of (1.6) is not verified). This is true if $f(e) = e$.

Theorem 2.6. If the function $f : G \to G$ has a fixed point, then the only solutions of equation (E_p) are the involutive automorphisms of n-group $(G,[\ ])$.

Proof: Choosing $e \in G$ a fixed point of f, from Theorem 2.5 we have $d = e$ and $f = g$. From Remark and [5] it follows that f is an morphism of n-group $(G,[\ ])$.

Conversely, if f is an automorphism of n-group with the property $f(f(x)) = x$, $x \in G$, then f verifies the equation (E_p).

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