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SOME FUNCTIONAL RELATIONS INVOLVING I-FUNCTION USING FRACTIONAL INTEGRAL OPERATORS

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Abstract. *There are many papers in mathematical literature in which the functional relation associated with the hypergeometric functions and the logarithmic derivative of gamma function (digamma function) is derived using the application of fractional integration operators. In order to unify and extend the existing results Katsuyuki Nishimoto and R. K. Saxena [16], R. K. Saxena [17], Kalla and Ross [13] and Kalla and Al-saquabi [12] have already contributed their observations. In the proposed paper we seek to establish functional relations between digamma functions and I-function by the application of Erdélyi-Kober integral operators.*

Keywords: *Gamma function, I-function, Erdélyi-Kober integral operator, Laplace transform, digamma function.*

Mathematics Subject Classification: *Primary 33C60, Secondary 33E50.*

1. INTRODUCTION

There are many integral operators of fractional calculus some of them are given below. Erdélyi-Kober operator is given by [21];

$$E_{0,x}^{\sigma,\eta} f(x) = \frac{x^{-\sigma-\eta}}{\Gamma\sigma} \int_0^x (x-t)^{\sigma-1} t^{\eta} f(t) dt \tag{1.1}$$

where $\operatorname{Re}(\sigma) \geq 0$, and

$$K_{0,x}^{\sigma,\eta} f(x) = J_{0,x}^{\sigma,\eta} f(x) = \frac{x^{\eta}}{\Gamma\sigma} \int_x^{\infty} (t-x)^{\sigma-1} t^{-\sigma-\eta} f(t) d \quad t \tag{1.2}$$

where $\operatorname{Re}(\sigma) > 0$.

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Various generalizations have been brought out from time to time, with a varying degree of importance given to physical problems. Mathematicians, like Kober [1], Erdėlyi [2], Saxena [18], Kalla and Saxena [3], Kalla[4], Nishimoto [8], Saxena and Kumbhat [5], Virginia S. Kiryakova [19], Mehdi Dalir, Majid Bashour [20], Jain, Jain and Thakur [22] and several others have worked in this field. A detailed account of the fractional integral operators with their applications can be found in papers written by Nishimoto [10, 15], Oldham and Spanier [6], Ross [7], McBride and Roach [11] and Samko et al. [14]. The object of the present paper is to establish functional relations between I-function and the logarithmic derivative of Gamma functions $\psi(x)$ which is known as digamma functions applying the application of Erdélyi-Kober integral operators.

2. THE I-FUNCTION

The I-function is the generalization of H-function which was introduced by Saxena [9], while solving a dual integral equation involving sum of H-functions as kernel. This is defined here:

$$I(x) = I_{p_i,q_i:r}^{m,n}[x] = I_{p_i,q_i:r}^{m,n}\left[x \left| \begin{matrix} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{n+1,q_i} \end{matrix} \right. \right] \tag{2.1}$$

$$= \frac{1}{2\pi i}\int_L \chi(s) x^s ds \tag{2.2}$$

where $x \neq 0$ and $x = \exp\{sLog|x| + i\arg x\}$ in which $Log|x|$ represents the natural logarithmic of $|x|$ and $\arg x$ is not necessarily the principal value. Here

$$\chi(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - B_j s)\prod_{j=1}^{n}\Gamma(1-a_j + A_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1-b_{ji} + B_{ji}s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji} - A_{ji}s)\right\}} \tag{2.3}$$

with all other conditions as already detailed in [18].

The following results [18] are required in this sequel.

$$\frac{x^{-\sigma-\eta}}{\Gamma\sigma}\int_0^x t^{\rho-1}(x-t)^{\sigma-1} t^{\eta} I_{p_i,q_i;r}^{m,n}\left[at \left| \begin{matrix} (a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{n+1,q_i} \end{matrix} \right. \right] dt =$$
$$= x^{\rho-1} I_{p_i+1,q_i+1;r}^{m,n+1}\left[ax \left| \begin{matrix} (1-\rho-\eta,1)(a_j, A_j)_{1,n}; (a_{ji}, A_{ji})_{n+1,p_i} \\ (b_j, B_j)_{1,m}; (b_{ji}, B_{ji})_{n+1,q_i}\ (1-\rho-\eta-\sigma,1) \end{matrix} \right. \right] \tag{2.4}$$

$$\frac{x^{\eta}}{\Gamma\sigma}\int_x^{\infty} t^{\rho-1}(t-x)^{\sigma-1} t^{-\sigma-\eta} I_{p_i,q_i;r}^{m,n}\left[at \left| \begin{matrix} (a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i} \\ (b_j,B_j)_{1,m};(b_{ji},B_{ji})_{n+1,q_i} \end{matrix} \right.\right] dt =$$
$$= x^{\rho-1} I_{p_i+1,q_i+1;r}^{m+1,n}\left[ax \left| \begin{matrix} (a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},(1-\rho-\eta,1) \\ (1-\rho-\eta-\sigma,1),(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{n+1,q_i} \end{matrix} \right.\right] \tag{2.5}$$

where:

$\mathrm{Re}\left(\rho+\min b_j/B_j\right)>0,\ j=1,\ 2,\ \ldots,\ m;\ \mathrm{Re}(\sigma)>0,\ |\arg a_i|<(1/2)\pi\lambda_i,$

$\lambda_i=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q_i}B_{ji}>0,$ where $i=1,\ 2,\ \ldots,\ r$ and

$\mu_i=\sum_{j=1}^{n}A_j+\sum_{j=n+1}^{p_i}A_{ji}-\sum_{j=1}^{m}B_j+\sum_{j=m+1}^{q_i}B_{ji}<0$ or $\mu_i=0;\ i=1,\ 2,\ \ldots,\ r$ and

$0<|at|<\beta_i^{-1};\ \beta_i=\prod_{j=1}^{p_i}(A_{ji})^{A_{ji}}\prod_{j=1}^{q_i}(B_{ji})^{-B_{ji}},\ i=1,\ 2,\ \ldots,\ r.$

Equations (2.4) and (2.5) find by using the following integral:

$$\int_0^x t^{\rho-1}(x-t)^{\sigma-1}dt = x^{\rho+\sigma-1}\frac{\Gamma\rho\Gamma\sigma}{\Gamma(\rho+\sigma)} \tag{2.6}$$

$$\int_x^{\infty} t^{\rho-1}(x-t)^{\sigma-1}dt = x^{\rho+\sigma-1}\frac{\Gamma(1-\rho-\sigma)\Gamma\sigma}{\Gamma(1-\rho)} \tag{2.7}$$

where, $\mathrm{Re}(\rho)>0$, $\mathrm{Re}(\sigma)>0$.

The computable form of I-function can be written as:

$$I_{p_i,q_i;r}^{m,n}[x]=\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v x^s \chi(s)}{v!B_h} \tag{2.8}$$

where $s=(b_h+v)/B_h$ exist for all $x\neq 0$ if $\mu_i<0$ and $0<|x|<\beta_i^{-1}$ if $\mu_i=0$, where $\mu_i=\sum_{j=1}^{n}A_j+\sum_{j=n+1}^{p_i}A_{ji}-\sum_{j=1}^{m}B_j+\sum_{j=m+1}^{q_i}B_{ji}$ and $\beta_i=\prod_{j=1}^{p_i}(A_{ji})^{A_{ji}}\prod_{j=1}^{q_i}(B_{ji})^{-B_{ji}}$, $i=1,\ 2,\ \ldots,\ r$. Here $\chi(s)$ is given by (2.3).

3. THE FUNCTIONAL RELATIONS

(I) In this section, we will establish the following relations by using Erdélyi-Kober operator:

$$\sum_{k=1}^{\infty}\frac{(\sigma)_k}{k} I_{p_i+1,q_i+1;r}^{m,n+1}\left[ax\left|\begin{matrix}(1-\rho-\eta-\sigma,1),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho-\eta-\sigma-k,1)\end{matrix}\right.\right]= \tag{3.1}$$

$$=\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s \chi(s)}{v!B_h}\{\psi(\rho+\eta+\sigma+s)-\psi(\rho+\eta+s)\}$$

where,
$s=(b_h+v)/B_h$, $\mathrm{Re}(\rho+\min b_j/B_j)>0$, $j=1,\ 2,\ \ldots,\ m$; $\mathrm{Re}(\sigma)\geq 0$, $|\arg a_i x|<(1/2)\pi\lambda_i$, $\lambda_i>0$; $\mu_i\leq 0$, $\lambda_i=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q_i}B_{ji}>0$, and $\mu_i=\sum_{j=1}^{n}A_j+\sum_{j=n+1}^{p_i}A_{ji}-\sum_{j=1}^{m}B_j+\sum_{j=m+1}^{q_i}B_{ji}<0$ or $\mu_i=0$; where $i=1,\ 2,\ \ldots,\ r$.

Proof: On differentiating both sides of (2.4) with respect to ρ according to Leibnitz's rule, it is found that

$$\frac{x^{-\sigma-\eta}}{\Gamma\sigma}\int_0^x t^{\rho+\eta-1}\ln t(x-t)^{\sigma-1} I_{p_i,q_i;r}^{m,n}\left[at\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]dt=$$

$$=x^{\rho-1}\left[\ln x\left[ax\left|\begin{matrix}(1-\rho-\eta,1),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho-\eta-\sigma,1)\end{matrix}\right.\right]+\right. \tag{3.2}$$

$$\left.+\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\chi(s)\Gamma(\rho+\eta+s)}{v!B_h\Gamma(\rho+\sigma+\eta+s)}\{\psi(\rho+\eta+s)-\psi(\rho+\eta+\sigma+s)\}\right]$$

(3.2)

where, $s=(b_h+v)/B_h$.

The R. H. S. of the equation (3.2) will be designated by $J(a_{p_i},A_{p_i};b_{q_i},B_{q_i};\rho,\sigma,\eta,a)$.

In term of Erdélyi-Kober operator, (3.2) can be written as:

$$E_{0,x}^{\sigma,\eta}f(x)=\frac{x^{-\sigma-\eta}}{\Gamma\sigma}\int_0^x(x-t)^{\sigma-1}t^{\eta}f(t)dt \tag{3.3}$$

On the account of the property of analyticity and continuity at $\sigma=0$ and $\eta=0$, we interchanging the role of η by $\eta-\sigma$ and then σ by $-\sigma$. Hence for the differentiation of

$$x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

to an arbitrary order, we find that

$$E_{0,x}^{-\sigma,\eta-\sigma}\left(x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}(ax)\right)=$$
$$=x^{\rho-1}\left[\ln xI_{p_i+1,q_i+1;r}^{m,n+1}\left[ax\left|\begin{matrix}(1-\rho-\eta-\sigma,1),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho-\eta,1)\end{matrix}\right.\right]+\right.$$
$$\left.+\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\chi(s)\Gamma(\rho+\eta+\sigma+s)}{v!B_h\Gamma(\rho+\eta+s)}\left\{\psi(\rho+\eta+\sigma+s)-\psi(\rho+\eta+s)\right\}\right] \tag{3.4}$$

Now we consider the following integral equation of Volterra type:

$$\frac{x^{-\sigma-\eta}}{\Gamma\sigma}\int_0^x(x-t)^{\sigma-1}t^{\eta}f(t)dt=x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}(ax) \tag{3.5}$$

where, $\mathrm{Re}\left(\rho+\min b_j/B_j\right)>0$, $\mathrm{Re}(\sigma)\geq 0$, $|\arg a_i|<(1/2)\pi\lambda_i$, $\lambda_i>0$; $\mu_i\leq 0$.

Since (3.5) is of convolution type, it can be solved by applying Laplace transform. However, we use the technique of fractional integration operator to solve it, due to its elegance and simplicity.

On writing (3.5) in the operator form, we have

$$E_{0,x}^{\sigma,\eta}f(x)=x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}(ax) \tag{3.6}$$

Operating on both side of (3.6) with $E_{0,x}^{-\sigma,\eta-\sigma}$

$$f(x)=E_{0,x}^{-\sigma,\eta-\sigma}\left(x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}(ax)\right) \tag{3.7}$$

In view of (3.4), we can write the solution of the integral (3.6) as

$$f(x)=x^{\rho-1}\left[\ln xI_{p_i+1,q_i+1;r}^{m,n+1}\left[ax\left|\begin{matrix}(1-\rho-\eta-\sigma,1),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho-\eta,1)\end{matrix}\right.\right]+\right.$$
$$\left.+\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\chi(s)\Gamma(\rho+\eta+\sigma+s)}{v!B_h\Gamma(\rho+\eta+s)}\left\{\psi(\rho+\eta+\sigma+s)-\psi(\rho+\eta+s)\right\}\right] \tag{3.8}$$

to verify the solution, we substitutes (3.8) in to (3.5) in terms of argument *t*.

On writing

$$t = x + t - x = x\left(1 + \frac{t-x}{x}\right) = x\left(1 - \frac{x-t}{x}\right)$$

where x and t are real and $x > 0$, so we obtain a series expansion of $\ln t$ in the form

$$\ln t = \ln x + \ln\left(1 + \frac{t-x}{x}\right) \tag{3.9}$$

when $\left|(t-x)/x\right| < 1$, $\ln\left(1 + \left((t-x)/x\right)\right)$ can be expanded in to a Taylor's series expansion. Thus

$$\ln t = \ln x - \sum_{k=1}^{\infty} \frac{(x-t)^k}{kx^k} \tag{3.10}$$

with the interval of the convergence $0 < t \le 2x$.

If we substitute (3.8) and (3.10) in (3.5) and evaluate the corresponding beta type integrals, the desired result (3.1) is achieved.

Special Cases:

If we put $\eta = 0$ and $\rho = \rho - \sigma$. Then this functional relation (3.1) reduces to well known functional relations of I-function by using Riemann Liouville operator [22]:

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{(\sigma)_k}{k} I^{m,n+1}_{p_i+1,q_i+1;r}\left[ax\left|\begin{matrix}(1-\rho,1),(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i},(1-\rho-k,1)\end{matrix}\right.\right] = \\&= \sum_{h=1}^{m}\sum_{v=0}^{\infty} \frac{(-1)^v (ax)^s \chi(s)}{v!B_h}\left\{\psi(\rho+s) - \psi(\rho-\sigma+s)\right\}\end{aligned} \tag{3.11}$$

If we put $p_i = p$, $q_i = q$ for all values of i and $r = 1$. Then functional relation (3.11) reduces to well known result [16] for H-function:

$$\begin{aligned}&\sum_{k=1}^{\infty} \frac{(\sigma)_k}{k} H^{m,n+1}_{p+1,q+1}\left[ax\left|\begin{matrix}(1-\rho,1),(a_j,A_j)_{1,n};(a_j,A_j)_{n+1,p}\\(b_j,B_j)_{1,m};(b_j,B_j)_{m+1,q},(1-\rho-k,1)\end{matrix}\right.\right] = \\&= \sum_{h=1}^{m}\sum_{v=0}^{\infty} \frac{(-1)^v (ax)^s \chi(s)}{v!B_h}\left\{\psi(\rho+s) - \psi(\rho-\sigma+s)\right\}\end{aligned} \tag{3.12}$$

(II) In this section, we establish the following functional relations by using Erdélyi-Kober operator:

$$\sum_{k=1}^{\infty}\frac{(-1)^k(\sigma)_k}{k}I^{m,n+1}_{p_i+1,q_i+1;r}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},(1-\rho+\eta,1)\\(1-\rho+\eta-k,1),(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]=$$
$$=\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\chi(s)}{v!B_h}\{\psi(1-\rho+\eta+\sigma-s)-\psi(1-\rho+\eta-s)\} \quad (3.13)$$

where
$s=(b_h+v)/B_h$, $\mathrm{Re}(\rho+\min b_j/B_j)>0$, $j=1,\ 2,\ \ldots,\ m$; $\mathrm{Re}(\sigma)\ge 0$, $|\arg a_i x|<(1/2)\pi\lambda_i$, $\lambda_i>0$; $\mu_i\le 0$. $\lambda_i=\sum_{j=1}^{n}A_j-\sum_{j=n+1}^{p_i}A_{ji}+\sum_{j=1}^{m}B_j-\sum_{j=m+1}^{q_i}B_{ji}>0$, and $\mu_i=\sum_{j=1}^{n}A_j+\sum_{j=n+1}^{p_i}A_{ji}-\sum_{j=1}^{m}B_j+\sum_{j=m+1}^{q_i}B_{ji}<0$ or $\mu_i=0$; $i=1,\ 2,\ \ldots,\ r$.

Proof: On differentiating both sides of (2.5) with respect to ρ according to Leibnitz's rule, it is found that

$$\frac{x^{\eta}}{\Gamma\sigma}\int_x^{\infty}t^{\rho-\sigma-\eta-1}\ln t\,(t-x)^{\sigma-1}I^{m,n}_{p_i,q_i;r}\left[at\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]dt=$$
$$=x^{\rho-1}\left[\ln x I^{m+1,n}_{p_i+1,q_i+1;r}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},(1-\rho+\eta+\sigma,1)\\(1-\rho+\eta,1),(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]+\right. \quad (3$$
$$\left.+\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\chi(s)\Gamma(1-\rho+\eta-s)}{v!B_h\Gamma(1-\rho+\eta+\sigma-s)}\{\psi(1-\rho+\eta-s)-\psi(1-\rho+\eta+\sigma-s)\}\right]$$

where $s=(b_h+v)/B_h$.

The R. H. S. of the (3.14) will be designated by $J\left(a_{p_i},A_{p_i};b_{q_i},B_{q_i};\rho,\eta,\sigma,a\right)$.

In term of Erdélyi-Kober operator, (3.14) can be written as

$$K^{\sigma,\eta}_{0,x}\left(x^{\rho-1}\ln x I^{m,n}_{p_i,q_i;r}(ax)\right)=J\left(a_{p_i},A_{p_i};b_{q_i},B_{q_i};\rho,\eta,\sigma,a\right) \quad (3.15)$$

On the account of the property of analyticity and continuity at $\sigma=0$ and $\eta=0$, we interchanging the role of η by $\eta-\sigma$ and then σ by $-\sigma$. Hence for the differentiation of

$$x^{\rho-1}\ln x I^{m,n}_{p_i,q_i;r}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]$$

to an arbitrary order, we find that

$$K_{0,x}^{-\sigma,\eta-\sigma}\left(x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}(ax)\right)=x^{\rho-1}\left[\ln xI_{p_i+1,q_i+1;r}^{m+1,n}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i}\\(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]+\right.$$
$$\left.+\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\chi(s)\Gamma(1-\rho+\eta+\sigma-s)}{v!B_h\Gamma(1-\rho+\eta-s)}\{\psi(1-\rho+\eta+\sigma-s)-\psi(1-\rho+\eta-s)\}\right]= \tag{3.16}$$
$$=J\left(a_{p_i},A_{p_i};b_{q_i},B_{q_i};\rho,\eta-\sigma,-\sigma,a\right)$$

where, $s=(b_h+v)/B_h$.

Now we consider the following Volterra type integral:

$$\frac{x^{\eta}}{\Gamma\sigma}\int_x^{\infty}(t-x)^{\sigma-1}t^{-\sigma-\eta}f(t)\,dt=x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}(ax) \tag{3.17}$$

where $\mathrm{Re}(\rho+\min b_j/B_j)>0$, $\mathrm{Re}(\sigma)\geq 0$, $|\arg a_i x|<(1/2)\pi\lambda_i$, $\lambda_i>0$; $\mu_i\leq 0$.

Since (3.17) is of convolution type, it can be solved by applying Laplace transform. However, we use the technique of fractional integration operator to solve it, due to its elegance and simplicity.

On writing (3.17) in the operator form, we have

$$K_{0,x}^{\sigma,\eta}f(x)=x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}(ax) \tag{3.18}$$

Operating on both sides of (3.18) $K_{0,x}^{-\sigma,\eta-\sigma}$ on both sides, we get

$$f(x)=K_{0,x}^{-\sigma,\eta-\sigma}\left(x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}(ax)\right) \tag{3.19}$$

In view of (3.16) we can write the solution of the integral (3.17) as

$$f(x)=\left[x^{\rho-1}\ln xI_{p_i,q_i;r}^{m,n}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},(1-\rho,1)\\(1-\rho+\eta+\sigma,1)(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]+\right.$$
$$\left.+\sum_{h=1}^{m}\sum_{v=0}^{\infty}\frac{(-1)^v(ax)^s\chi(s)\Gamma(1-\rho+\eta+\sigma-s)}{v!B_h\Gamma(1-\rho+\eta-s)}\{\psi(1-\rho+\eta+\sigma-s)-\psi(1-\rho+\eta-s)\}\right] \tag{3.20}$$

To verified the solution, we substitutes (3.20) in to (3.17) in terms of argument *t*.

$$t=x+t=x\left(1+\frac{t-x}{x}\right)=x\left(1-\frac{x-t}{x}\right)$$

where *x* and *t* are real and $x>0$, so we obtain a series expansion of $\ln t$ in the form

(3.21)

$$\ln t = \ln x + \ln\left(1+\frac{t-x}{x}\right)$$

(3.21)

when $\left|\frac{t-x}{x}\right| < 1, \ln\left(1+\frac{t-x}{x}\right)$ can be expanded in to a Taylor's series expansion. Thus

$$\ln t = \ln x - \sum_{k=1}^{\infty} \frac{(x-t)^k}{kx^k} \tag{3.22}$$

with the interval of the convergence $0 < t \le 2x$.

If we substitutes (3.20) and (3.22) in (3.17) and evaluate the corresponding beta type integrals, the desired result (3.13) is achieved.

Special Cases:

If we putting $\eta = 0$ Then functional relation (3.13) reduces to well known functional relations of I-function by using Weyl integral operator [22].

$$\sum_{k=1}^{\infty} \frac{(\sigma)_k}{k} I_{p_i+1,q_i+1}^{m+1,n}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p_i},(1-\rho,1)\\(1-\rho-k,1),(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q_i}\end{matrix}\right.\right]= \tag{3.23}$$
$$= \sum_{h=1}^{m}\sum_{v=0}^{\infty} \frac{(-1)^v (ax)^s \chi(s)}{v!B_h}\{\psi(1-\rho+\sigma-s)-\psi(1-\rho-s)\}$$

If we put $p_i = p$, $q_i = q$ for all values of i and $r=1$ then functional relation (3.23) reduces to well known result [16] for H-function:

$$\sum_{k=1}^{\infty} \frac{(\sigma)_k}{k} H_{p+1,q+1}^{m+1,n}\left[ax\left|\begin{matrix}(a_j,A_j)_{1,n};(a_{ji},A_{ji})_{n+1,p},(1-\rho,1)\\(1-\rho-k,1),(b_j,B_j)_{1,m};(b_{ji},B_{ji})_{m+1,q}\end{matrix}\right.\right]= \tag{3.24}$$
$$= \sum_{h=1}^{m}\sum_{v=0}^{\infty} \frac{(-1)^v (ax)^s \chi(s)}{v!B_h}\{\psi(\rho+s)-\psi(\rho-\sigma+s)\}$$

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