*ORIGINAL PAPER*

# ON THE MONOTONY OF $\left(1+\frac{1}{n}\right)^{n+0.5}$ AND AN APPLICATION

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***Abstract.*** *In this paper we expose two different proofs for the fact that the sequence of general term* $(1+1/n)^{n+0.5}$ *which converges to the celebrated constant of Napier also called the number of Euler, the number e, is strictly decreasing. The one of the proofs shows the possibility of work without to use the differential calculus, the other is probably the most natural. We recall the situation of the monotony in the general case of* $(1+1/n)^{n+p}$, *we point out the order of convergence of* $(1+/n)^{n+0.5}$ *and we show how the monotony of the considered sequence can be used to obtain the shortest proof of the weak form of Stirling's formul.*

***Keywords:*** *Sequence, monotony, the number e, Stirling's formula.*

## 1. INTRODUCTION

In a recent paper [4], published in the American Mathematical Monthly, are exposed three proofs for the inequality

$$e < \left(1+\frac{1}{n}\right)^{n+0.5}. \qquad (1.1)$$

Now, we propose to give an elementary method to establish that the sequence $(x_n)_n$ of general term

$$x_n = \left(1+\frac{1}{n}\right)^{n+0.5}, \qquad (1.2)$$

which tends to *e*, is strictly decreasing. On this way, (1.1) will follow again, as an immediate consequence; this proof will may be added to the proofs of [4]. Our method will involve no any tools of calculus, as derivatives, Taylor series expansions, or integrals, but only the notion of limit of a sequence, because the number *e* is, by one of its (equivalent) definitions, a limit.

In the final part, we will use (1.1) to give a "shortcut" in the proof of the weak form of Stirling's formula

$$\lim_{n\to\infty} \frac{n!}{n^n e^{-n}\sqrt{2\pi n}} = 1. \qquad (1.3)$$

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## 2. THE MONOTONY OF THE SEQUENCE $(x_n)_n$

By a simple calculation, we have

$$\frac{x_n}{x_{n+1}} = \left(1 + \frac{1}{n(n+2)}\right)^n \cdot \frac{n+1}{n+2} \cdot \frac{n+1}{\sqrt{n(n+2)}}. \quad (2.1)$$

To prove that the ratio $x_n / x_{n+1}$ is greater than 1, we will use a generalization of the inequality of Bernoulli, namely

$$(1+x)^n > 1 + nx + \frac{n(n-1)}{2} x^2, \qquad (n \in \mathbf{N},\ n \ge 3,\ x > 0) \quad (2.2)$$

which contains in the right part only the first three terms of the finite Newton's expansion of $(1 + x)^n$ and follows from this, if we do a minorizing, keeping only these three terms. (It also can be obtained by a very simple induction, respecting to $n$.) Applying (2.2) for $x = 1/n(n+2)$, we obtain

$$\left(1 + \frac{1}{n(n+2)}\right)^n > \frac{2n^3 + 10n^2 + 13n - 1}{2n(n+2)^2}. \quad (2.3)$$

Use now the $GM - AM$ inequality for the two numbers $a = 1$ and $b = n(n+2)/(n+1)^2$; we have

$$\sqrt{\frac{n(n+2)}{(n+1)^2}} < \frac{1}{2}\left(1 + \frac{n(n+2)}{(n+1)^2}\right)$$

that gives

$$\frac{n+1}{\sqrt{n(n+2)}} > \frac{2(n+1)^2}{2n^2 + 4n + 1}. \quad (2.4)$$

Minorizing the first and the third factor of the right part of (2.1), by (2.3), respectively by (2.4) and performing the necessary calculation, we obtain

$$\frac{x_n}{x_{n+1}} > \frac{4n^6 + 32n^5 + 38n^4 + 140n^3 + 92n^2 + 2n - 2}{4n^6 + 32n^5 + 38n^4 + 140n^3 + 88n^2 + 16n}$$

then $x_n/x_{n+1} > 1$, i.e. the sequence is strictly decreasing for any $n$ such that $92n^2 + 2n - 2 \ge 88n^2 + 16n$, that is $n \ge 4$; for $n = 1, 2, 3$, the situation can be seen directly. ■

Note that this proof is different to the one of [9].

## 3. A LITTLE ADDENDUM: A DIFFERENTIAL PROOF

A standard differential proof of the inequality $x_n > x_{n+1}$ is also possible. Indeed, consider the function

$$f:(0,\infty)\to(0,\infty),\ \ f(x)=(1+1/x)^{x+0.5},$$

which extends the sequence $(x_n)_n$. The function $f$ has the same monotony as its logarithm; having log $f(x) = (x + 0.5)(\log(x+1) - \log x)$, we obtain, by a little calculation

$$\left(\log f\left(x\right)\right)'=\log\left(x+1\right)-\log x-\frac{1}{2}\left(\frac{1}{x+1}+\frac{1}{x}\right) \tag{2.5}$$

and the right part is negative because of the right part of the inequality of Hermite-Hadamard, applied to the convex function $t\mapsto \varphi(t) = 1/t$, $t > 0$, on the interval $[x, x+1] \subset (0, \infty)$, namely

$$\frac{2}{2x+1}<\log\left(x+1\right)-\log x<\frac{1}{2}\left(\frac{1}{x}+\frac{1}{x+1}\right)$$

[8, 4]. Then $f$ is strictly decreasing, therefore $x_n > x_{n+1}$. ■

The negativity of $(\log f(x))'$ also can be deduced from the combination of the relations $(\log f(x))''(x) = 1/2x^2(x+1)^2 > 0$ and $\lim_{x\to\infty} (\log f(x))' = 0$).

## 4. TWO REMARKS CONCERNING THE SEQUENCE $(x_n)_n$

Considering the family of sequences of general term $(1 + 1/n)^{n+p}$, where $p \in \mathbf{R}$, a such sequence is strictly decreasing if and only if $p \geq 0.5$ [9]; moreover, for $p < 0.5$, the sequence is strictly increasing, beginning from a rank which depends of $p$. Indeed, passing to the positive real variable, we find quicker as in [9] these results, by using the function $\varphi_p: (0, \infty)\to (0, \infty)$, $\varphi_p(x) = (1 + 1/x)^{x+p}$, because of the relations

$$\left(\log\varphi_p\left(x\right)\right)'=\log\left(1+\frac{1}{x}\right)-\frac{x+p}{x\left(x+1\right)};\ \ \lim_{x\to\infty}\left(\log\varphi_p\left(x\right)\right)'=0;\ \ \left(\log\varphi_p\left(x\right)\right)''=\frac{\left(2p-1\right)x+p}{x^2\left(x+1\right)^2}.$$

Also, as a second remark, see that, for $p = 0.5$ and only for this value, the first iterated limit of the sequence of general term $(1 + 1/n)^{n+p}$ is equal to zero

$$\lim_{x\to\infty} n\left(\left(1+\frac{1}{n}\right)^{n+0.5}-e\right)=0$$

and, for the second one, we have

$$\lim_{x\to\infty} n^2\left(\left(1+\frac{1}{n}\right)^{n+0.5}-e\right)=\frac{e}{12}$$

These both results can be obtained passing to the real variable $x \in (0,\infty)$ and using the rule of L'Hospital, or, with a deeper explanation, putting $p = 0.5$ in the limited expansion

$$\left(1+\frac{1}{n}\right)^{n+p} = e + \left(p-\frac{1}{2}\right)\frac{e}{n} + \frac{12p^2-24p+11}{24}\frac{e}{n^2} + O\left(\frac{1}{n^3}\right).$$

So, we see that the speed of convergence of the sequence $(x_n)_n$ is of the order of $1/n^2$.

## 5. AN APPLICATION OF (1.1): A "SHORTCUT" IN THE PROOF OF THE WEAK FORM OF STIRLING'S FORMULA

The formula (1.3) is so called because it is the simplest one that gives an idea of the magnitude of $n!$; there are many more advanced formulas than (1.3), namely two-sided estimates of $n!$ [6] and the references therein and moreover, a more complete description is given by the Stirling asymptotic series for $n!$ [1, 7].

One of the most used way to obtain (1.3) consists in the following two steps:

*(I) It is established that the sequence of general term*

$$a_n = \frac{n!}{n^n e^{-n}\sqrt{n}} = \frac{n!}{n^{n+0.5}e^{-n}} \tag{4.1}$$

is convergent to a limit $a \neq 0$.

*(II) It is used the fact that, if a sequence $(a_n)_n$ converges to a limit $a \neq 0$, then*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{a_n^2}{a_{2n}}. \tag{4. 2}$$

Applying (4.2) for the sequence (4.1) and using the formula of Wallis, it is established that $a = \sqrt{2\pi}$ [2, 5].

Our "shortcut" in the proof is related especially to the part (I).

We have

$$\frac{a_{n+1}}{a_n} = \frac{e}{\left(1+\frac{1}{n}\right)^{n+0.5}} < 1,$$

because of the inequality (1.1). Then the sequence $(a_n)_n$ is strictly decreasing.

Then, to obtain that the sequence is convergent, it is sufficient to find that it is lower bounded. Let

$$\Omega_n = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots 2n} \tag{4.3}$$

be. We have

$$\Omega_n < \frac{1}{\sqrt{2n+1}}. \tag{4.4}$$

Before to continue, note that it exists more accurate inequalities concerning $\Omega_n$, namely two sided-estimates, e.g.

$$\frac{1}{\sqrt{\pi(n+1/2)}} < \Omega_n < \frac{1}{\sqrt{\pi n}} \tag{4.5}$$

which contains in the extreme parts, under the square root, the same constant, that is $\pi$. [6, 3]. A quick proof of (4.5) can be obtained from the known double inequality $I_{2n+2} < I_{2n+1} < I_{2n}$ , where $I_n$ are the so-called integrals of Wallis, $I_n = \int_0^{\pi/2} \sin^n x dx$, by using the equalities $I_{2n} = \Omega_n \pi/2$ and $I_{2n+1} = 1/\Omega_n(2n+1)$.)

In our proof of the part (I) it will be sufficient to use only the rough and ready inequality (4.4). Let $(u_n)_n$ be a strictly increasing sequence, $u_n > 0$, for any $n \in$ **N.** Then the following inequality holds

$$u_n > \frac{u_n^2}{u_{2n}}. \tag{4.6}$$

This relation is reminiscent of the equality (4.2); but it contains no any limit and it is an ine-quality, in place of a equality. Let now

$$u_n = \frac{e^n n!}{n^n}$$

be. From the relation

$$\frac{u_{n+1}}{u_n} = \frac{e}{\left(1+\frac{1}{n}\right)^n} > 1$$

we see that the sequence $(u_n)_n$ is strictly increasing. Apply to it the inequality (4.6). By an elementary calculation, we obtain

$$\frac{u_n^2}{u_{2n}} = \frac{1}{\Omega_n}, \tag{4.7}$$

that gives, because of (4.6), $u_n > 1/\Omega_n$, which conducts us to

$$\frac{e^n n!}{n^n} > \frac{1}{\Omega_n}$$

and, from (4.4), we have, by transitivity, $u_n > \sqrt{2n+1}$, that is

$$\frac{e^n n!}{n^n} > \sqrt{2n+1}. \tag{4.8}$$

This allows us to obtain that

$$\frac{e^n n!}{n^n\sqrt{n}} > \frac{\sqrt{2n+1}}{\sqrt{n}} > \sqrt{2}\ ,$$

so the sequence $(a_n)_n$ is convergent to a limit $a$ and, moreover, $a > 0$, then $a \neq 0$. ■

This proof was did improving an idea of [11]. The length of the present proof may be compared with the one of the usual proofs of textbooks.

## 6. TWO FINAL REMARKS

Note that the inequality (4.8) also allows us to find, in an elementary manner (that is, in this case: without to use the formula of Stirling!), the limit for $n \to\infty$ of the unique nontrivial sequence of general term $u_n(\alpha) = \alpha^n n! \,/\, n^n$, $\alpha > 1$, namely the one for $\alpha = e$. This limit is $\infty$.

In the spirit of our proof of the step (I), based only on the use of $\Omega_n$, mention that we can quickly do the step (II), also by using this $\Omega_n$, as follows: we have $a_n = u_n / \sqrt{n}$ . So

$$\frac{a_n^2}{a_{2n}} = \left(\frac{u_n}{\sqrt{n}}\right)^2 \frac{\sqrt{2n}}{u_{2n}} = \frac{u_n^2}{u_{2n}}\sqrt{\frac{2}{n}} = \frac{1}{\Omega_n}\sqrt{\frac{2}{n}} \to \sqrt{2\pi}\ ,$$

because of $\lim_{n\to\infty} \Omega_n \sqrt{n} = 1/\sqrt{\pi}$ , as a direct consequence of (4.5). So $\lim_{n\to\infty} a_n = \sqrt{2\pi}$ .

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