

EXTENDED FRACTIONAL FOURIER TRANSFORM OF DISTRIBUTIONS OF COMPACT SUPPORT

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Abstract. *Extended fractional Fourier transform which is generalization of fractional Fourier transform with two more parameters and, is defined on the space of generalized functions. Its analyticity is established. Also the inversion of the transform is obtained along with uniqueness.*

Keywords: *Extended fractional Fourier transform, fractional Fourier transform, Inversion and Uniqueness.*

1. INTRODUCTION

The fractional Fourier transform (FrFT) is the generalization of Fourier transform (FT) with angular parameter α . Namias [4] introduced it with the help of eigen values as,

$$\left[F^\alpha \left[f(t) \right] \right] (u) = \frac{e^{i \left[\frac{\pi - 2\alpha}{4} \right]}}{\sqrt{2\pi \sin \alpha}} \int_{-\infty}^{\infty} e^{-i \left[\frac{(t^2 + u^2)}{2} \cot \alpha - tu \csc \alpha \right]} f(t) dt \quad (1.1)$$

Mathematically it is α_{th} ordered Fourier transform. Number of applications of (1.1) can be found in [5].

McBride and Kerr [3] had shown the necessity of modifying the definition of Namias and provided a mathematical frame work for it. He extended (1.1) on the spaces of generalized functions δ . Later on Bhosale and Chaudhary in [1] extended FrFT to the distributions of compact support.

Further generalization of FrFT can be seen in Hua [2] with two more parameters as,

$$\left[F_{a,b}^\alpha \left[f(t) \right] \right] (u) = F_{a,b}^\alpha (u) = \int_{-\infty}^{\infty} e^{i\pi \left[(a^2 t^2 + b^2 u^2) \cot \alpha - 2abtu \csc \alpha \right]} f(t) dt \quad (1.2)$$

Note that for $a = b = \sqrt{\frac{-1}{2\pi}}$, it is FrFT.

In this paper we have proved the properties of kernel and Inverse of extended fractional Fourier transform in section 2. Analyticity is proved in section 3 and lastly Inversion theorem and Uniqueness is presented in section 4.

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2. KERNEL PROPERTIES AND INVERSION FORMULA

In this section we have proved some properties of kernel and obtained its inversion formula.

2.1. PROPERTIES OF KERNEL

The following properties of kernel $K_{a,b}^\alpha(t, u)$ are simple to prove.

- i) $K_{a,b}^\alpha(t, u) = K_{b,a}^\alpha(u, t)$
- ii) $K_{a,b}^{-\alpha}(t, u) = K_{a,b}^{\alpha*}(t, u)$ where $*$ denotes conjugate.
- iii) $K_{a,b}^\alpha(-t, u) = K_{a,b}^\alpha(t, -u)$
- iv) $\int K_{a,b}^\alpha(t, u) K_{b,c}^\beta(u, v) = \sqrt{\frac{-1}{ib^2(\cot \alpha + \cot \beta)}} K_{a,c}^{\alpha+\beta}(t, v)$

For practical use of the transform its inverse plays an important role therefore next we have obtained the inverse of extended fractional Fourier transform.

2.2. INVERSION FORMULA

If $K_{a,b}^\alpha(u)$ is extended FrFT of $f(t)$ as in (1.2) then it is possible to recover the function f by means of the Inversion formula,

$$f(t) = ab \csc \alpha \int_{-\infty}^{\infty} K_{a,b}^\alpha(u) K_{a,b}^{-\alpha}(t, u) du \quad (2.1)$$

where $K_{a,b}^{-\alpha}(t, u) = e^{-i\pi(a^2t^2+b^2u^2)\cot \alpha + 2iabtu \csc \alpha}$.

Proof: The extended fractional Fourier transform given by (1.2) is,

$$\left[F_{a,b}^\alpha[f(t)] \right](u) = F_{a,b}^\alpha(u) = \int_{-\infty}^{\infty} K_{a,b}^\alpha(t, u) f(t) dt$$

where the kernel $K_{a,b}^\alpha(t, u) = e^{i\pi(a^2t^2+b^2u^2)\cot \alpha - 2iabtu \csc \alpha}$

$$\begin{aligned} F_{a,b}^\alpha(u) &= \int_{-\infty}^{\infty} f(t) e^{i\pi(a^2t^2+b^2u^2)\cot \alpha - 2iabtu \csc \alpha} dt \\ e^{-i\pi b^2u^2 \cot \alpha} F_{a,b}^\alpha(u) &= \int_{-\infty}^{\infty} f(t) e^{i\pi a^2t^2 \cot \alpha - 2iabtu \csc \alpha} dt \\ &= \int_{-\infty}^{\infty} f(t) e^{i\pi a^2t^2 \cot \alpha} e^{-i\pi 2abtu \csc \alpha} dt \\ &= \int_{-\infty}^{\infty} g(t) e^{-i\pi 2abtu \csc \alpha} dt \end{aligned}$$

where $g(t) = f(t)e^{i\pi a^2 t^2 \cot \alpha}$ (say)

$$= [F(g(t))](2\pi ab u \csc \alpha)$$

$$= [F(g(t))](\eta) \quad (2.2)$$

where F denotes ordinary Fourier transform and $\eta = 2\pi ab u \csc \alpha$ (say) (2.2) becomes,

$$e^{-i\pi b^2 \cot \alpha \left(\frac{\eta}{2\pi ab \csc \alpha}\right)^2} F_{a,b}^\alpha \left(\frac{\eta}{2\pi ab \csc \alpha}\right) = [F(g(t))](\eta) = G(\eta) \quad (\text{say}) \quad (2.3)$$

Then invoking Fourier inversion, we can also write as

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\eta) e^{i\eta t} d\eta \quad (2.4)$$

Substituting $g(t)$, $G(\eta)$ from (2.2) and (2.3) in (2.4)

$$\begin{aligned} f(t)e^{i\pi a^2 t^2 \cot \alpha} &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{a,b}^\alpha \left(\frac{\eta}{2\pi ab \csc \alpha}\right) e^{-i\pi b^2 \cot \alpha \left(\frac{\eta}{2\pi ab \csc \alpha}\right)^2} e^{i\eta t} d\eta \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} F_{a,b}^\alpha(u) e^{-i\pi b^2 u^2 \cot \alpha} e^{i2\pi ab u \csc \alpha} 2\pi ab \csc \alpha du \\ f(t) &= ab \csc \alpha \int_{-\infty}^{\infty} e^{-i\pi b^2 u^2 \cot \alpha} e^{-i\pi a^2 t^2 \cot \alpha} e^{2\pi i ab t u \csc \alpha} F_{a,b}^\alpha(u) du \\ &= ab \csc \alpha \int_{-\infty}^{\infty} e^{-i\pi \left[a^2 t^2 + b^2 u^2\right] \cot \alpha - 2\pi ab t u \csc \alpha} F_{a,b}^\alpha(u) du \\ &= ab \csc \alpha \int_{-\infty}^{\infty} K_{a,b}^{-\alpha}(t, u) F_{a,b}^\alpha(u) du \end{aligned}$$

where $K_{a,b}^{-\alpha}(t, u) = e^{-i\pi(a^2 t^2 + b^2 u^2) \cot \alpha + 2\pi ab t u \csc \alpha}$.

3. EXTENDED FRFT ON THE SPACES OF COMPACT SUPPORT AND ITS ANALYTICITY

Next the extended fractional Fourier transform is defined on the space of generalized functions and its analyticity is proved.

3.1. TESTING FUNCTION SPACE E

Definition 3.1. The test function space E

An infinitely differentiable complex valued function φ on \mathcal{R}^n belongs to $E(\mathcal{R}^n)$ if for each compact set $K \subset S_c$ where $S_c = \{t : t \in \mathcal{R}^n, |t| \leq c, c > 0\}$, $K \in \mathcal{R}^n$

$$\gamma_{E,K}(\varphi) = \sup_{t \in K} |D^K \varphi(t)| < \infty$$

In what follows $E(\mathcal{R}^n)$ will denote the space of all $\varphi \in E(\mathcal{R}^n)$ with support contained in S_c .

Note that the space E is complete therefore a Frechet space.

It can be easily proved that $K_{a,b}^\alpha(t, v) \in E$ for $0 \leq \alpha < \frac{\pi}{2}$ and $v \in \mathcal{R}^n$.

3.2. EXTENDED FrFT DEFINED ON E :

For $0 \leq \alpha < \frac{\pi}{2}$ and $v \in \mathcal{R}^n$, the extended fractional Fourier transform of $f \in E^1(\mathcal{R}^n)$ can be defined by,

$$[F_{a,b}^\alpha[f(t)]](v) = F_{a,b}^\alpha(v) = \langle f(t), K_{a,b}^\alpha(t, v) \rangle \quad (3.1)$$

where right-hand side has a meaning as the application of $f \in E^1$ to $K_{a,b}^\alpha(t, v) \in E$.

It can be extended to the complex space as an entire function given by,

$$[F_{a,b}^\alpha[f(t)]](u) = F_{a,b}^\alpha(u) = \langle f(t), K_{a,b}^\alpha(t, u) \rangle \quad (3.2)$$

The right-side is meaningful because for each $u \in C^n$, $K_{a,b}^\alpha(t, u) \in E$, as a function of t .

Next we prove the analyticity of extended fractional Fourier transform.

3.3. ANALYTICITY THEOREM

Theorem: Let $f \in E^1(\mathcal{R}^n)$ and let its extended fractional Fourier transform be defined by (3.2). Then $F_{a,b}^\alpha(u)$ is analytic on C^n if the supp $f \subset S_c = \{t : t \in \mathcal{R}^n, |t| \leq c, c > 0\}$ and for each $\varepsilon > 0$, there exist the constants c_1 and a positive integer k such that for $0 \leq \alpha < \frac{\pi}{2}$

$$|F_{a,b}^\alpha(u)| \leq c_1 c_\alpha^k [2a^2(c + \varepsilon) \cos \alpha + 2ab |\operatorname{Im} u|]^k \\ \exp \left[c_\alpha \left[(a^2(c + \varepsilon)^2 + b^2 |\operatorname{Im} u|^2) \cos \alpha - 2ab(c + \varepsilon) |\operatorname{Im} u| \right] \right]$$

Moreover $F_{a,b}^\alpha(u)$ is differentiable and

$$D_u^k F_{a,b}^\alpha(u) = \left\langle f(t), D_u^k K_{a,b}^\alpha(t, u) \right\rangle$$

Proof: Let $u = (u_1, u_2, \dots, u_j, \dots, u_n) \in C^n$.

We first prove that

$$\frac{\partial}{\partial u} F_{a,b}^\alpha(u) = \left\langle f(t), \frac{\partial}{\partial u} K_{a,b}^\alpha(t, u) \right\rangle$$

For fixed $u_j \neq 0$, choose two concentric circles C and C^1 with center at u_j and radii r and r_1 respectively, such that $0 < r < r_1 < |u_j|$. Let Δu_j be a complex increment satisfying $0 < |\Delta u_j| < r$.

Consider

$$\frac{F_{a,b}^\alpha(u_j + \Delta u_j) - F_{a,b}^\alpha(u_j)}{\Delta u_j} - \left\langle f(t), \frac{\partial}{\partial u} K_{a,b}^\alpha(t, u) \right\rangle = \left\langle f(t), \varphi_{\Delta u_j}(t) \right\rangle \quad (3.3)$$

where $\varphi_{\Delta u_j}(t) = \frac{i}{\Delta u_j} \left[K_{a,b}^\alpha(t, u, \dots, u_j + \Delta u_j, \dots, u_n) - K_{a,b}^\alpha(t, u) \right] - \frac{\partial}{\partial u} K_{a,b}^\alpha(t, u)$.

For any fixed $t \in \mathcal{R}^n$ and any fixed integer $k = (k_1, k_2, \dots, k_n) \in \mathcal{N}^n$,

$$\begin{aligned} D_t^k F_{a,b}^\alpha(t, u) &= D_t^k e^{i\pi[(a^2 t^2 + b^2 u^2) \cot \alpha - 2abtu \csc \alpha]} \\ &= \sum_{m \leq k} \binom{k}{m} (2i\pi \csc \alpha)^{k-m} (a^2 \cos \alpha)^m {}^k C_m (a^2 t \cos \alpha - abu)^{k-2m} K_{a,b}^\alpha(t, u) \\ &= \sum_{m \leq k} {}^k C_m C_\alpha (a^2 t \cos \alpha - abu)^{k-2m} K_{a,b}^\alpha(t, u) \end{aligned}$$

where $C_\alpha, {}^k C_m$ are the constants depending on α, k and m .

Since for any fixed $t \in \mathcal{R}^n$, fixed integer k and α ranging from 0 to $\frac{\pi}{2}$, $D_t^k F_{a,b}^\alpha(t, u)$ is analytic inside and on C^1 . We have by Cauchy integral formula,

$$\begin{aligned} D_t^k \varphi_{\Delta u_j}(t) &= D_t^k \int_{C^1} K_{a,b}^\alpha(t, \tilde{u}) \left(\frac{1}{\Delta u_j} \left(\frac{1}{z - u_j - \Delta u_j} - \frac{1}{z - u_j} \right) - \frac{1}{(z - u_j)^2} \right) dz \\ &= \Delta u_j \int_{C^1} \frac{D_t^k K_{a,b}^\alpha(t, \tilde{u})}{(z - u_j - \Delta u_j)(z - u_j)^2} dz \end{aligned}$$

where $\tilde{u} = (u_1, u_2, \dots, u_{j-1}, z, u_{j+1}, \dots, u_n)$

$$= \Delta u_j \int_{C^1} \frac{M(t, \tilde{u})}{(z - u_j - \Delta u_j)(z - u_j)^2} dz$$

But for all $z \in C_1$ and t restricted to a compact subset of \mathcal{R}^n , $0 \leq \alpha < \frac{\pi}{2}$, $M(t, \tilde{u}) = D_t^k K_{a,b}^\alpha(t, \tilde{u})$ is bounded by a constant k .

$$\therefore |D_t^k \varphi_{\Delta u_j}(t)| \leq |\Delta u_j| K / (r_1 - r) r_1$$

Thus, as $|\Delta u_j| \rightarrow 0$, $D_t^k \varphi_{\Delta u_j}(t)$ tends to zero uniformly on the compact subset of \mathcal{R}^n , therefore it follows that $\varphi_{\Delta u_j}(t)$ converges in $E(\mathcal{R}^n)$ to zero. Since $f \in E^1$ we conclude that (3.3) also tends to zero. Therefore, $F_{a,b}^\alpha(u)$ is differentiable with respect to u_j . But this is true for all $j = 1, 2, 3, \dots, n$ hence $F_{a,b}^\alpha(u)$ is analytic on C^n , and

$$D_u^k F_{a,b}^\alpha(u) = \langle f(t), D_u^k K_{a,b}^\alpha(t, u) \rangle$$

To prove the second part suppose that $\text{supp } f \subset C$.

Then, by the boundedness property of the generalized functions, there exists a constant c and a non-negative integer k , such that

$$\begin{aligned} |F_{a,b}^\alpha(u)| &= \left| \langle f(t), K_{a,b}^\alpha(t, u) \rangle \right| \\ &\leq C \max_{|\beta| \leq k} \sup_{t \in \mathcal{R}^n} |D_t^\beta K_{a,b}^\alpha(t, u)| \\ &\leq C \max_{|\beta| \leq k} \left| 2\pi a^2 t \cot \alpha - 2\pi abu \csc \alpha \right|^\beta \sup_{t \in \mathcal{R}^n} \exp \left(i\pi \left[(a^2 t^2 + b^2 u^2) \cot \alpha - 2abtu \csc \alpha \right] \right) \\ &\leq C \max_{|\beta| \leq k} \left| 2\pi a^2 t \cot \alpha - 2\pi abu \csc \alpha \right|^\beta \sup_{t \in \mathcal{R}^n} \exp \left(i\pi \left[(a^2 t^2 + b^2 u^2) \cot \alpha - 2abtu \right] \right) \\ &\leq c_1 c_\alpha^k \left(2a^2 (c + \varepsilon) \cos \alpha + 2ab |\text{Im } u| \right)^k \\ &\quad \exp \left[c_\alpha \left[\left(a^2 (c + \varepsilon)^2 + b^2 |\text{Im } u|^2 \right) \cos \alpha - 2ab (c + \varepsilon) |\text{Im } u| \right] \right] \end{aligned}$$

where $c_\alpha = \pi \csc \alpha$ for constant c_1 and c_α .

4. INVERSION THEOREM

In this section we have proved the inverse theorem in the generalized sense as in [6].

Theorem 4.1: Let $f \in F^1(\mathcal{R})$, $0 \leq \alpha < \frac{\pi}{2}$ and $\text{supp } f \subset S_c : S_c = \{t : t \in \mathcal{R}^n, |t| \leq c, c > 0\}$ and let $F_{a,b}^\alpha(u)$ be the distributional extended fractional Fourier transform of f as defined by

$$F_{a,b}^\alpha[f(t)](u) = F_{a,b}^\alpha(u) = \langle f(t), K_{a,b}^\alpha(t, u) \rangle$$

where $K_{a,b}^\alpha(t, u) = e^{i\pi[(a^2t^2 + b^2u^2)\cot\alpha - 2abtu\csc\alpha]}$ that is, $F_{a,b}^\alpha(u) = \int_{-\infty}^{\infty} e^{i\pi[(a^2t^2 + b^2u^2)\cot\alpha - 2abtu\csc\alpha]} f(t) dt$.

Then for each $\varphi \in D(\mathcal{R})$ we have,

$$\langle f(t), \varphi(t) \rangle = \lim_{N \rightarrow \infty} \left\langle ab \csc \alpha \int_{-N}^N K_{a,b}^{-\alpha}(t, u) F_{a,b}^\alpha(u) du, \varphi(t) \right\rangle$$

Proof: To prove the inversion theorem, we state the following two lemmas.

Lemma 1: Let $F_{a,b}^\alpha[f(t)](u) = F_{a,b}^\alpha(u)$ for $0 \leq \alpha < \frac{\pi}{2}$, and $\text{supp } f \subset S_c : S_c = \{t : t \in \mathcal{R}^n, |t| \leq c, c > 0\}$, for $\varphi(t) \in D(\mathcal{R})$

$$\Phi(u) = \int_{-\infty}^{\infty} K_{a,b}^{-\alpha}(t, u) \varphi(t) dt$$

Then for any fixed number $-\infty < r < \infty$

$$\int_{-r}^r \Phi(u) \langle f(v), K_{a,b}^\alpha(v, u) \rangle d\tau = \left\langle f(v), \int_{-r}^r \Phi(u) K_{a,b}^\alpha(v, u) d\tau \right\rangle, \quad (u = \sigma + i\tau) \quad (4.1)$$

where $u \in C^n$ and v is restricted to a compact subset of \mathcal{R} .

Proof: The case $\varphi(t) = 0$ is trivial, so that we consider $\varphi(t) \neq 0$, it can be easily seen that

$$\int_{-r}^r \Phi(u) \langle K_{a,b}^\alpha(v, u) d\tau \rangle, \quad (u = \sigma + i\tau)$$

is a C^∞ function of v and it belongs to E .

Hence the right side of (4.1) is meaningful.

To prove the equality, we construct the Riemann-sum for this integral and write

$$\begin{aligned} \int_{-r}^r \langle f(v), K_{a,b}^\alpha(v, u) \rangle \Phi(u) d\tau &= \lim_{m \rightarrow \infty} \sum_{n=-m}^{m-1} \langle f(v), K_{a,b}^\alpha(v, \sigma + i\tau_{n,m}) \rangle \Phi(\sigma + i\tau_{n,m}) \Delta\tau_{n,m} \\ &= \lim_{m \rightarrow \infty} \left\langle f(v), \sum_{n=-m}^{m-1} K_{a,b}^\alpha(v, \sigma + i\tau_{n,m}) \Phi(\sigma + i\tau_{n,m}) \Delta\tau_{n,m} \right\rangle \end{aligned}$$

Taking the operator D_v^k within the integral and summation signs, which is easily justified as,

$$\begin{aligned} &\gamma_{K,k} \left\{ \sum_{n=-m}^{m-1} K_{a,b}^\alpha(v, \sigma + i\tau_{n,m}) \Phi(\sigma + i\tau_{n,m}) \Delta\tau_{n,m} - \int_{-r}^r \Phi(u) K_{a,b}^\alpha(v, u) d\tau \right\} \\ &= \sup_{v \in K} \left\{ \sum_{n=-m}^{m-1} D_v^k K_{a,b}^\alpha(v, \sigma + i\tau_{n,m}) \Phi(\sigma + i\tau_{n,m}) \Delta\tau_{n,m} - \int_{-r}^r \Phi(u) K_{a,b}^\alpha(v, u) d\tau \right\} \\ &\lim_{m \rightarrow \infty} \sum_{n=-m}^{m-1} D_v^k K_{a,b}^\alpha(v, \sigma + i\tau_{n,m}) \Phi(\sigma + i\tau_{n,m}) \Delta\tau_{n,m} = \int_{-r}^r \Phi(u) K_{a,b}^\alpha(v, u) d\tau \end{aligned}$$

where $v \in K$.

It follows that for every m , the summation is a member of E and it converges in E to the right side of (4.1). Hence proof.

Lemma 2: For $\varphi(t) \in D(I)$, set $\Phi(u)$ as in lemma (1) above for $u \in C, v$, restricted to the subset of \mathcal{R} , then

$$M_r(v) = \int_{-r}^r \Phi(u) K_{a,b}^\alpha(v, u) d\tau = \int_{-r}^r K_{a,b}^\alpha(v, u) d\tau \int_{-\infty}^{\infty} \varphi(t) K_{a,b}^{-\alpha}(t, u) d\tau \quad (4.2)$$

converges in E to $\varphi(v)$ as $r \rightarrow \infty$.

Proof: We shall show that $M_r(v) \rightarrow \varphi(v)$ in E as $r \rightarrow \infty$.

It is to show

$$\gamma_{K,k} [M_r(v) - \varphi(v)] = \sup_{v \in K} |D_v^k [M_r(v) - \varphi(v)]| \rightarrow 0 \text{ as } r \rightarrow \infty$$

Prove that for $k = 0$

$$\int_{-\infty}^{\infty} K_{a,b}^\alpha(t, u) \int_{-\infty}^{\infty} \varphi(v) K_{a,b}^{-\alpha}(t, u) d\tau = \varphi(v)$$

It is to say that $\lim_{m \rightarrow \infty} M_r(v) = \varphi(v)$.

Since the integrand is a C^∞ function of v and $\varphi \in D(\mathcal{R})$, we can repeatedly differentiate to the integral sign in (4.2) and integrals are uniformly convergent, we have

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} D_u^k K_{a,b}^{\alpha}(t, u) \int_{-\infty}^{\infty} \varphi(v) K_{a,b}^{-\alpha}(t, u) d\tau = \varphi(v)$$

Hence the claim.

Let $\varphi(t) \in D(\mathcal{R})$ we shall show that.

$$\left\langle \int_{-r}^r K_{a,b}^{-\alpha}(t, u) F_{a,b}^{\alpha}(u) d\tau, \varphi(t) \right\rangle \text{ trends to } \langle f(t), \varphi(t) \rangle \text{ as } r \rightarrow \infty \quad (4.3)$$

From the analyticity of $F_{a,b}^{\alpha}(u)$ on C and the fact that $\varphi(t)$ has compact support in \mathcal{R} , it follows that the left side expression in (4.3) is merely a repeated integral with respect to t and u and the integral in (4.3) is a continuous function of t as the closed bounded domain of the integration.

Therefore, we can write (4.3) as

$$\begin{aligned} \int_{-\infty}^{\infty} \varphi(t) \int_{-r}^r K_{a,b}^{-\alpha}(t, u) F_{a,b}^{\alpha}(u) d\tau dt &= \int_{-\infty}^{\infty} \varphi(t) \int_{-r}^r K_{a,b}^{-\alpha}(t, u) \langle f(v), K_{a,b}^{\alpha}(v, u) \rangle d\tau dt \\ &= \int_{-r}^r \langle f(v), K_{a,b}^{\alpha}(v, u) \rangle \int_{-\infty}^{\infty} \varphi(t) K_{a,b}^{-\alpha}(t, u) dt d\tau \end{aligned}$$

Since $\varphi(t)$ is compact support, and the integral is a continuous function of (t, u) the order of integration may be changed. The change in the order of integration is justified by appeal to lemma (1).

This yields,

$$\int_{-r}^r \langle f(v), K_{a,b}^{\alpha}(v, u) \rangle \Phi(u) d\tau$$

where $\Phi(u)$ is as in lemma (1).

$$\text{This is equal to } \int_{-r}^r \langle f(v), K_{a,b}^{-\alpha}(v, u) \rangle \Phi(u) d\tau \quad (4.4)$$

Again by lemma (2), equation (4.4) converges to $\langle f(t), \varphi(t) \rangle$ as $r \rightarrow \infty$.

Hence the theorem.

Theorem 4.2: (Uniqueness):

If $[R_{a,b}^{\alpha}[f(t)]](u) = F_{a,b}^{\alpha}(u)$ and $[R_{a,b}^{\alpha}[g(t)]](u) = G_{a,b}^{\alpha}(u)$ for $0 \leq \alpha < \frac{\pi}{2}$ and $\text{supp } f \subset S_c : S_c = \{t : t \in \mathcal{R}, |t| \leq c\}$ and $\text{supp } g \subset S_c : S_c = \{t : t \in \mathcal{R}, |t| \leq c\}$ if $F_{a,b}^{\alpha}(u) = G_{a,b}^{\alpha}(u)$ then $f = g$ in the sense of equality in $D^1(I)$.

Proof: Proof is simple hence omitted.

CONCLUSION

In this paper we proved the Inversion formula of extended FrFT. It will be useful in the future findings of extended FrFT. Analytic, Inversion and Uniqueness theorems of extended FrFT are also proved which will be useful in partial differential equations.

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