ORIGINAL PAPER

THE PEXIDER VERSION OF A FUNCTIONAL EQUATION RELATED TO POMPEIU'S AND HOSSZÚ EQUATIONS

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Abstract. *In [8] and [9] we showed that Pompeiu's equation and Hosszú's equation are particular cases of a large class of functional equations related by Cauchy kernel with respect to two binary operations. In this paper by using the operations* $x \circ y = x + y + xy$ *and* $x * y = x + y - xy$ *, which defines Pompeiu and Hosszú equations, we study a pexiderized version of them, i.e.* $f(x + y + xy) + g(x + y - xy) = h(x) + k(y)$.
Keywords: *functional equation, Pompeiu's equation, Hosszu's equation.*

1. INTRODUCTION

On the set of real numbers we consider the binary operations

$$x \circ y = x + y + xy \text{ and } x * y = x + y - xy,\ x, y \in \mathbf{R},$$

which are associative and commutative.

Moreover, the groups $(\mathbf{R} \setminus \{-1\}, 0)$ and $(\mathbf{R} \setminus \{1\}, *)$ are isomorphic with the multiplicative group $(\mathbf{R}^*, \cdot)$ and the isomorhisms are defined by

$$\varphi : \mathbf{R} \setminus \{-1\} \to \mathbf{R}^*,\ \varphi(x) = x + 1,\ x \neq 1 \text{ and}$$
$$\psi : \mathbf{R}^* \to \mathbf{R} \setminus \{1\},\ \psi(x) = 1 - x,\ x \neq 0.$$

The function $\psi \circ \varphi : \mathbf{R} \setminus \{-1\} \to \mathbf{R} \setminus \{1\}$ is also an isomorphism.

Definition 1.1. The group $(\mathbf{R} \setminus \{-1\}, \circ)$ is called the Pompeiu's group and the functiona equation of its morphisms

(1) $\quad f(x \circ y) = f(x) \circ f(y),\ x, y \in \mathbf{R}$

is called Pompeiu's equation.

Remark 1.1. Using the isomorphism φ it follows that the solutions of Pompeiu's equation are the functions of the form

$$f(x) = M(x + 1) - 1,\ x \in \mathbf{R},$$

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where $M:\mathbf{R}\to\mathbf{R}$ is a multiplicative function $(M(x\cdot y)=M(x)\cdot M(y),\ x,y\in\mathbf{R})$.

Another equation related by Pompeiu's group was proposed by Yugoslavy to 21^{th} I.M.O. London:

(2) $$f(x+y+xy)=f(x)+f(y)+f(xy),\quad x,y\in\mathbf{R}.$$

In [1] is proved that this equation is equivalent with Cauchy equation, the solutions are additive function.

Definition 1.2. The functional equation $f:\mathbf{R}\to\mathbf{R}$

(3) $$f(x+y-xy)+f(xy)=f(x)+f(y),\quad x,y\in\mathbf{R}$$

is called Hosszú's functional equation.

In [2] and [11] is prove that this equation is equivalent with Jensen equation, the solutions are of the form

$$f(x)=A(x)+a,\quad x\in\mathbf{R},$$

where $A:\mathbf{R}\to\mathbf{R}$ is additive function and $a\in\mathbf{R}$ is a constant.

Remark 1.2. The equation:

(4) $$f(x+y-xy)=f(x)+f(y)-f(x)\cdot f(y),\ x,y\in\mathbf{R}$$

or

$$f(x*y)=f(x)*f(y),\quad x,y\in\mathbf{R}$$

is the equation of the morphism of semigroup $(\mathbf{R},*)$ and using the isomorphism ψ these solutions are of the form

$$f(x)=1-M(1-x),\quad x,y\in\mathbf{R},$$

where $M:\mathbf{R}\to\mathbf{R}$ is a multiplicative function.

Comparing the deviation of a function $f:\mathbf{R}\to\mathbf{R}$ from a morphism of the semigroup $(\mathbf{R},\circ)$ with the deviation from the morphism of the semigroup $(\mathbf{R},*)$ in [10] we solved the equation:

(5) $$f(x+y+xy)+f(x+y-xy)=2(f(x)+f(y)),\quad x,y\in\mathbf{R}.$$

The equation (5) are equivalent of Cauchy equation and its solutions are additive functions.

Remark 1.3. a) if we denote

$$P(f)(x,y)=f(x\circ y)-f(x)\circ f(y)$$

the deviation from a morphism of the function f in the semigroup $(\mathbf{R},\circ)$ and by

$$H(f)(x,y) = f(x*y) - f(x)*f(y)$$

the deviation of the function f from the morphism of the group $(\mathbf{R},*)$, the equation (5) can be rewritten in the form

(5.1) $$P(f)(x,y) + H(f)(x,y) = 0\,, \quad x,y \in \mathbf{R}$$

or

(5.2) $$f(x\circ y) + f(x*y) = f(x)\circ f(y) + f(x)*f(y)\,, \quad x,y \in \mathbf{R}\,.$$

b) The equation (5) is Problem 2 from the International Contest: The Clock-Tower School, 2011 and its solution can be found in [10].

c) The Pexider versions of Pompeiu's equation can be found in [4], [7] and the Pexider version of Hosszú's equation can be found in [3], [5] and [6].

Our goal is to solve a pexiderized version of the equation (5):

(6) $$f(x+y+xy) + g(x+y-xy) = h(x) + k(y)$$

or

(6.1) $$f(x\circ y) + g(x*y) = h(x) + k(y)\,.$$

2. MAIN RESULTS

We consider the functional equation

(6) $$f(x+y+xy) + g(x+y-xy) = h(x) + k(y)\,, \quad x,y \in \mathbf{R}\,,$$

where $f,g,h,k : \mathbf{R} \to \mathbf{R}$ are unknown functions.

Theorem 2.1. *If the functions* f,g,h,k *verifies the equation* (6) *then we have:*

(7) $$h(x) = f(x) + g(x) - k(0)\,, \quad x \in \mathbf{R}$$

(8) $$k(y) = f(y) + g(y) - h(0)\,, \quad x \in \mathbf{R}$$

(9) $$g(x) = f(2x+1) - f(x) - f(1) + g(0) + f(0)\,, \quad x \in \mathbf{R}$$

and the function f *verifies the equation*

(10) $$f(x+y+xy) + f(2x+2y-2xy+1) - f(x+y-xy) = f(2x+1) + f(2y+1)\,, \quad x,y \in \mathbf{R}\,.$$

Proof: If in (6) we put $y=0$ and next $x=0$ we obtain (7) and (8).
Using (7) and (8) in (6) it follows:

(11) $$f(x+y+xy)+g(x+y-xy)=f(x)+f(y)+g(x)+g(y)-h(0)-k(0).$$

Taking in (11) $x=y=0$ it follows

$$f(0)+g(0)=h(0)+k(0)$$

and then for $y=1$ we obtain (9).
Replacing the expression of g from (9) in (11) we obtain the relation (10).

Theorem 2.2. *The equation* (10) *is equivalent with the equation:*

(12) $$f(x+y+xy)+f(x+y-xy+2)=f(2x+1)+f(2y+1),\quad x,y\in\mathbf{R}.$$

Proof: The associativity of the operation $*$ leads to

$$(x*y)*(-1)=x*(y*(-1)).$$

From (10) we substitute $y:=2y-1$ and we obtain

(13) $$f(2xy+2y-1)+f(-4xy+4x+4y-1)-f(-2xy+2x+2y-1)$$
$$=f(2x+1)+f(4y-1)-f(1),\quad x,y\in\mathbf{R}.$$

In (10) we substitute $x:=x+y-xy$, $y:=-1$ and we obtain:

(14) $$f(-1)+f(-4xy+4x+4y-1)-f(-2xy+2x+2y-1)$$
$$=f(2x+2y-2xy+1)+f(-1)-f(1),\quad x,y\in\mathbf{R}.$$

Subtracting the relations (13) and (14) we obtain:

(15) $$f(2xy+2y-1)+f(2x+2y-2xy+1)=f(2x+1)+f(4y-1),\quad x,y\in\mathbf{R}.$$

If in (15) we replace $2y$ by y we obtain:

(16) $$f(xy+y-1)+f(2x+y-xy+1)=f(2x+1)+f(2y-1),\quad x,y\in\mathbf{R}.$$

Now replacing y by $y+1$ we obtain (12).

Theorem 2.3. *The function* $f:\mathbf{R}\to\mathbf{R}$ *verifies the equation* (12) *if and only if the function* $H:\mathbf{R}\to\mathbf{R}$, $H(x)=f(-4x-3)$, $x\in\mathbf{R}$ *verifies the Hosszú's equation:*

$$H(x+y-xy)+H(xy)=H(x)+H(y),\quad x,y\in\mathbf{R}.$$

Proof: Replacing in (12) x by $x+1$ and y by $y+1$ we obtain

(17) $$f(2x+2y+xy+3)+f(-xy+3)=f(2x+3)+f(2y+3),\quad x,y\in\mathbf{R}.$$

We define the function $u:\mathbf{R}\to\mathbf{R}$,

$$u(x)=f(x+3)\quad\text{or}\quad f(x)=u(x-3),\quad x\in\mathbf{R}$$

and from (17) the function u satisfies the equation

(18) $$u(2x+2y+xy)+u(-xy)=u(2x)+u(2y),\quad x,y\in\mathbf{R}.$$

If in (18) we replace x by $2x$ and y by $2y$ we obtain:

(19) $$u(-4x-4y+4xy)+u(-4xy)=u(-4x)+u(-4y),\quad x,y\in\mathbf{R}.$$

From (19) the function $H:\mathbf{R}\to\mathbf{R}$ defined by

$$H(x)=u(-4x),\quad x\in\mathbf{R}$$

satisfies the equation:

(20) $$H(x+y-xy)+H(xy)=H(x)+H(y),\quad x,y\in\mathbf{R},$$

which is the Hosszú's equation.

Theorem 2.4. *The functions* $f,g,h,k:\mathbf{R}\to\mathbf{R}$ *verifies the equation* (6) *if and only if there exist an additive function* $A:\mathbf{R}\to\mathbf{R}$ *and the constants* $a,b,c,d\in\mathbf{R}$ *such that* $a+b=c+d$ *and* $f(x)=A(x)+a$, $g(x)=A(x)+b$, $h(x)=2A(x)+c$, $k(x)=2A(x)+d$, $x\in\mathbf{R}$.

Proof: Using the general solution of the equation (20) we have $H(x)=A_1(x)+a_1$, $x\in\mathbf{R}$, where $A_1:\mathbf{R}\to\mathbf{R}$ is additive and $a_1\in\mathbf{R}$ is a constant. Thus

$$u(x)=H\left(-\frac{1}{4}x\right)=A_1\left(-\frac{1}{4}x\right)+a_1=-\frac{1}{4}A_1(x)+a_1=A(x)+a_1,$$

where $A:\mathbf{R}\to\mathbf{R}$ is additive. Finally we have

$$f(x)=u(x-3)=A(x-3)+a_1=A(x)+A(-3)+a_1=A(x)+a,\quad x\in\mathbf{R}.$$

From Theorem 2.1 we obtain:

$$\begin{aligned}g(x)&=A(2x+1)+a-A(x)-a-f(1)+g(0)+f(1)\\&=2A(x)+A(1)-A(x)+g(0)=A(x)+b,\quad x\in\mathbf{R}\end{aligned}$$

$$h(x) = 2A(x) + c\ , \quad x \in \mathbf{R}\ ,$$

$$k(x) = 2A(x) + d\ , \quad x \in \mathbf{R}\ ,$$

where a, b, c, d are real constants and from (6) we obtain the condition $a + b = c + d$.

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