ORIGINAL PAPER

SOME NEW IDENTITIES ON THE CONIC SECTIONS

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Abstract. *In mathematics, a conic section (or just conic) is a curve obtained as the intersection of a cone (more precisely, a right circular conical surface) with a plane. In this paper, we construct some new identities and proposed the concept of the power of a point with respect to a conic.*

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1. THE ECCENTRICITY OF CONIC SECTION

Definition 1.1. A *conic section* (or conic) is a curve in which, a plane, not passing through the cone's vertex, intersects a cone.

Conics possess a number of properties, one of them consisting in the following result.

Proposition 1.2. [2] Each conic section, except for a circle, is a plane locus of points the ratio of whose distances from a fixed point F and a fixed line d is constant. The point F is called the focus of conic, the line d its directrix.

Proof: Let (ℓ) be the curve in which the plane (P) intersects a cone. We inscribe a sphere in the cone, which touches the plane (P) at the point F. Let (ω) be the plane containing the circle along which the sphere touches the cone. We take an arbitrary point $M \in (\ell)$ and draw through it a generator of the cone, and denote by B the point of its intersection with the plane (ω). We then drop a perpendicular from M to the line d of intrsection of the planes (P) and (ω), example: $MA \perp d$. We obtain $FM = BM$ because they are the tangents to the sphere drawn from one point. Further, if we denote by h the distance of M from the plane (ω), then $AM = \frac{h}{\sin\alpha}$, $BM = \frac{h}{\sin\beta}$, where α is the angle between the planes (ω) and (P) and β is the angle between the generator of the cone and the (ω). Hence it follows that $\frac{AM}{FM} = \frac{AM}{BM} = \frac{\sin\beta}{\sin\alpha}$.

Thus, the ratio $\lambda = \frac{AM}{FM} = \frac{\sin\beta}{\sin\alpha}$ does not depend on the point M. □

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We note that if $\lambda < 1$then (ℓ) is an *ellipse*; if $\lambda = 1$ then (ℓ) is a *parabola* and if $\lambda > 1$ then (ℓ) is a *hypebol*. The number λ is called the *eccentricity* of the conic section.

Let us now pass over to rectangular Cartesian coordinates Oxy in the plane (P), where $F(0,0)$ and $d: x = p$. Suppose that $M(x, y)$. Then $AM = \lambda FM$ if and only if $(1-\lambda^2)x^2 - 2p\lambda^2 x + y^2 - p^2\lambda^2 = 0.$

(1) If $\lambda = 1$ and by putting $\frac{p}{2} - x$ by x then we obtain the canonical equation of the parabola $(P): y^2 = 2px$ and $F\Big(\frac{p}{2}, 0\Big), d: x = -\frac{p}{2}.$

(2) If $\lambda < 1$ then $(1-\lambda^2)\Big(x + \frac{p\lambda^2}{1-\lambda^2}\Big)^2 + y^2 = \frac{p^2\lambda^2}{1-\lambda^2}$. By putting for brevity $x + \frac{p\lambda^2}{1-\lambda^2}$ by x and $a^2 = \frac{p^2\lambda^2}{(1-\lambda^2)^2}$, $b^2 = \frac{p^2\lambda^2}{1-\lambda^2}$ we get the canonical equation for the ellipse $(E): \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and $F\Big(\frac{p\lambda^2}{1-\lambda^2}, 0\Big), d: x = p + \frac{p\lambda^2}{1-\lambda^2} = \frac{p}{1-\lambda^2}.$

(3) If $\lambda > 1$ then $(1-\lambda^2)\Big(x + \frac{p\lambda^2}{1-\lambda^2}\Big)^2 + y^2 = \frac{p^2\lambda^2}{1-\lambda^2}$. By putting for brevity $x + \frac{p\lambda^2}{1-\lambda^2}$ by x and $a^2 = \frac{p^2\lambda^2}{(1-\lambda^2)^2}$, $-b^2 = \frac{p^2\lambda^2}{1-\lambda^2}$ we get the canonical equation for the hypebol $(H): \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $F\Big(\frac{p\lambda^2}{1-\lambda^2}, 0\Big), d: x = \frac{p}{1-\lambda^2}.$ □

2. PARAMETRIZATION AND POWER

Proposition 2.1. The parabol $(P): y^2 = 4px, p \neq 0,$ is parameterized by $\begin{cases} x = pt^2 \\ y = 2pt. \end{cases}$. Let the stright line $d: x = k(y - v) + u$ be passed through by the point $N(u, v)$ which intersected (P) at the points A and B. With the point $M(1, k)$ belong to the stright line $x = 1$ we obtain the identity $\frac{\overline{NA}.\overline{NB}}{OM^2} = v^2 - 4pu.$ The ratio $\frac{\overline{NA}.\overline{NB}}{OM^2}$ is called the {\rm power} of the point N with respect to parabol (P).

Proof: The coordinates of A, B are the solutions of consider

$$\begin{cases} x = k(y - v) + u \\ 4px = y^2. \end{cases}$$

Consider the system of equations:

$$\begin{cases} x-u=k(y-v) \\ y^2-4pky+4pkv-4pu=0. \end{cases}$$

Let y_1, y_2 are the solutions of equation $y^2-4pky+4pkv-4pu=0$. Inaddition, we have $A(x_1=k(y_1-v)+u, y_1)$, $B(x_2=k(y_2-v)+u, y_2)$.

Thus, we have $\dfrac{NA.NB}{OM^2}=|(y_1-v)(y_2-v)|=|v^2-4pu|$.

Hence, we have identities $\dfrac{\overline{NA}.\overline{NB}}{OM^2}=v^2-4pu$. Because of $I(p,0)$, the power of a focus point I with respect to the Parabol (P) is $-4p^2$. □

Exercise 2.2. Constructing power lines of two parabols.

Proposition 2.3. The circle $(C): x^2+y^2=1$ is a rational planar graphs in □, parameterized by $x(t)=\dfrac{2t}{1+t^2}, y(t)=\dfrac{1-t^2}{1+t^2}$, provided that $x(\infty)=\lim\limits_{t\to\infty}\dfrac{2t}{1+t^2}=0$, $y(\infty)=\lim\limits_{t\to\infty}\dfrac{1-t^2}{1+t^2}=-1$. The equation of the tangent line At to (C) at a point $A(x_0,y_0)\in(C)$ is $xx_0+yy_0=1$.

Proof: Equation of a line (d) through point $(0;1)\in(C)$ with slope $-t$ is $(d): y=-tx+1$. The line (d) meet (C) at the points $(0;1)$ and $A_t\Big(\dfrac{2t}{1+t^2},\dfrac{1-t^2}{1+t^2}\Big)$.

The point $\Big(\dfrac{2t}{1+t^2},\dfrac{1-t^2}{1+t^2}\Big)$ could be anywhere in (C) except $(0;-1)$. Provided that $x(\infty)=\lim\limits_{t\to\infty}\dfrac{2t}{1+t^2}=0, y(\infty)=\lim\limits_{t\to\infty}\dfrac{1-t^2}{1+t^2}=-1$, A_t could be $(0;-1)$.

Due to $A(x_0,y_0)\in(C)$, we have $x_0^2+y_0^2=1$.

Thus, $At: 2x_0(x-x_0)+2y_0(y-y_0)=0$, in other words, $At: xx_0+yy_0=1$. □

Proposition 2.4. Given $(C): x^2+y^2=R^2$. Provided that $N(u,v)$ and a line with slope $k=\tan\alpha$ through point N, meet (C) at A and B. Consider the point $M(R\sin\alpha, R\cos\alpha)$ in (C), we have $\overline{NA}.\overline{NB}=u^2+v^2-R^2$, in other words, $\overline{NA}.\overline{NB}.OM^2=R^4\Big(\dfrac{u^2}{R^2}+\dfrac{v^2}{R^2}-1\Big)$. We say that $\overline{NA}.\overline{NB}.OM^2$ is the *power of a point* N with respect to circle (C), review Proposition 2.6.

Proof: We could easily obtain this result. □

Proposition 2.5. The ellip $(E):\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ is parameterized by

$$x(t)=\frac{2a^2bt}{b^2+a^2t^2},\, y(t)=\frac{b^3-a^2bt^2}{b^2+a^2t^2}$$

with convention

$$\begin{cases} x(\infty)=\lim\limits_{t\to\infty}\dfrac{2a^2bt}{b^2+a^2t^2}=0 \\ y(\infty)=\lim\limits_{t\to\infty}\dfrac{b^3-a^2bt^2}{b^2+a^2t^2}=-b. \end{cases}$$

Proof: The line (d) through $(0;b)\in(E)$ with slope $-t$: $(d): y=-tx+b.$ (d) meets (E) at $(0;b)$ and $A_t\Big(\dfrac{2a^2bt}{b^2+a^2t^2},\dfrac{b^3-a^2bt^2}{b^2+a^2t^2}\Big).$ The point $A_t\Big(\dfrac{2a^2bt}{b^2+a^2t^2},\dfrac{b^3-a^2bt^2}{b^2+a^2t^2}\Big)$ through all points of (E), exept $(0;-b)$. With convention that $x(\infty)=\lim\limits_{t\to\infty}\dfrac{2a^2bt}{b^2+a^2t^2}=0, y(\infty)=\lim\limits_{t\to\infty}\dfrac{b^3-a^2bt^2}{b^2+a^2t^2}=-b$ deduce A_t through $(0;-b)$. \par\noindent Since $A(x_0,y_0)\in(E)$ we have $\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}=1.$
Hence $At:\dfrac{2x_0}{a^2}(x-x_0)+\dfrac{2y_0}{b^2}(y-y_0)=0$ or $At:\dfrac{xx_0}{a^2}+\dfrac{yy_0}{b^2}=1.$ □

Proposition 2.6. The ellip $(E):\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ with $a,b\in\mathbb{R}$ and the focus points $F(c,0)$, $F'(-c,0)$. Construct a line through the focus point F, meet ellip (E) at A and B. We have

(1) $\dfrac{1}{FA}+\dfrac{1}{FB}=\dfrac{2a}{b^2}.$

(2) Minimum value of AB is $\dfrac{2b^2}{a}.$

(3) Assuming $N(u,v)$ and a line through N with slope $k=\tan\alpha$ meet (E) at C and D. With the point $M(a\sin\alpha,b\cos\alpha)\in(E)$ we have $\overline{NC}.\overline{ND}.OM^2=a^2b^2\Big(\dfrac{u^2}{a^2}+\dfrac{v^2}{b^2}-1\Big).$

We call $\overline{NC}.\overline{ND}.OM^2$ is *the power* of the point N respect to ellip (E).

(4) The power of the focus point F respect to ellip (E) $\overline{FA}.\overline{FB}.OM^2$ is $-b^4$.

Proof: (1) We calculate $T = \dfrac{1}{FA} + \dfrac{1}{FB}$. Assuming $\alpha = \angle xFA \le \dfrac{\pi}{2}$, $r = FA$ and $A(x_1, y_1)$. Draw $AP \perp Ox$. We have $FP = r\cos\alpha$ and x-axis of A: $x_1 = c + r\cos\alpha$. Thus, we have

$$\begin{cases} FA + F'A = 2a \\ FA^2 - F'A^2 = (x_1 - c)^2 + y_1^2 - (x_1 + c)^2 - y_1^2 = -4cx_1. \end{cases}$$

We deduce

$$\begin{cases} r + F'A = 2a \\ r - F'A = -2\dfrac{cx_1}{a} \end{cases} \text{ and } \quad r = a - \dfrac{cx_1}{a}.$$

We have $ar = a^2 - cx_1 = a^2 - c(c + r\cos\alpha)$ or $FA = r = \dfrac{b^2}{a + c\cos\alpha}$.

Similary, we have $FB = \dfrac{b^2}{a - c\cos\alpha}$. We deduce $T = \dfrac{1}{FA} + \dfrac{1}{FB} = \dfrac{2a}{b^2}$.

(2) From $AB = FA + FB = \dfrac{b^2}{a + c\cos\alpha} + \dfrac{b^2}{a - c\cos\alpha} = \dfrac{2ab^2}{a^2 - c^2\cos^2\alpha}$ we deduce $AB \ge \dfrac{2ab^2}{a^2} = \dfrac{2b^2}{a}$. Thus, the minimum value of AB is $\dfrac{2b^2}{a}$, equality holds if $\alpha = \dfrac{\pi}{2}$ or $FA \perp Ox$.

(3) The Equation $NC: y = k(x - u) + v$ or $y = kx + h$ with $h = v - ku$. Coodinates of C and D are solutions of $\begin{cases} y = kx + h \\ \dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1. \end{cases}$

Let x_1, x_2 are two solutions of equation $(b^2 + a^2k^2)x^2 + 2hka^2x + a^2h^2 - a^2b^2 = 0$.

Thus, we have $C(x_1, y_1 = k(x_1 - u) + v)$ and $D(x_2, y_2 = k(x_2 - u) + v)$.

Deduce $NC.ND = |(u - x_1)(u - x_2)|(1 + k^2)$.

From $(b^2 + a^2k^2)x^2 + 2hka^2x + a^2h^2 - a^2b^2 = (b^2 + a^2k^2)(x - x_1)(x - x_2)$ we have $(b^2 + a^2k^2)(u - x_1)(u - x_2) = b^2u^2 + a^2v^2 - a^2b^2$.

Deduce $NC.ND = |(u - x_1)(u - x_2)|(1 + k^2) = |b^2u^2 + a^2v^2 - a^2b^2| \dfrac{1 + k^2}{b^2 + a^2k^2}$.

Conclude that $\overline{NC}.\overline{ND}(b^2\cos^2\alpha + a^2\sin^2\alpha) = a^2b^2\Big(\dfrac{u^2}{a^2} + \dfrac{v^2}{b^2} - 1\Big)$.

(4) From $c^2 = a^2 - b^2$, deduce $\overline{FA}.\overline{FB}.OM^2 = a^2b^2\Big(\dfrac{c^2}{a^2} + \dfrac{0^2}{b^2} - 1\Big) = -b^4$. □

Exercise 2.7. Constructing power line of two ellipse.

Proposition 2.8. Hypebol (H): $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with $a,b\in\mathbb{Q}$ is a rational planar graphs in $\mathbb{Q}$, parameterized by $x(t)=\frac{a+ab^2t^2}{1-b^2t^2}, y(t)=\frac{2b^2t}{1-b^2t^2}$, convention that

$$x(\infty)=\lim_{t\to\infty}\frac{a+ab^2t^2}{1-b^2t^2}=-a,\ \ y(\infty)=\lim_{t\to\infty}\frac{2b^2t}{1-b^2t^2}=0.$$

Proof: The line (d) through $(a;0)\in(H)$ with slope at: (d):x=a(ty+1). (d) meets (H) at $(a;0)$ and $A_t\Big(\frac{a+ab^2t^2}{1-b^2t^2},\frac{2b^2t}{1-b^2t^2}\Big)$. The point $A_t\Big(\frac{a+ab^2t^2}{1-b^2t^2},\frac{2b^2t}{1-b^2t^2}\Big)$ through all points of (H), exept $(-a;0)$. With convention that

$$x(\infty)=\lim_{t\to\infty}\frac{a+ab^2t^2}{1-b^2t^2}=-a, y(\infty)=\lim_{t\to\infty}\frac{2b^2t}{1-b^2t^2}=0$$

deduce A_t through $(-a;0)$. □

Proposition 2.9. Hypebol $(H):\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with $a,b\in\mathbb{R}$ and the focus points $F(c,0),\ F'(-c,0)$. Consider:

(1) The line d with slope k through the focus point F_1 meet Hypebol (H) at A and B. With $M\Big(\frac{ab}{\sqrt{|b^2-a^2k^2|}},\frac{abk}{\sqrt{|b^2-a^2k^2|}}\Big)\in(H)$ we have $\frac{\overline{F_1A}.\overline{F_1B}}{OM^2}=-\frac{b^2}{a^2}$, called *power of the point* F_1 with respect to (H).

(2) Suppose that $N(u,v)$ and a line with slope k through N meet (H) at C and D. We have $\frac{\overline{NC}.\overline{ND}}{OM^2}=a^2b^2\Big(\frac{u^2}{a^2}-\frac{v^2}{b^2}-1\Big)$. The ratio $\frac{\overline{NC}.\overline{ND}}{OM^2}$ is called the *power* of the point N relatvie to hypebol (H).

Proof: (1) From the focus point F_1 of hypebol H, construct a line $d: y=k(x-c)$ meet (H) at A and B. We have the coordinates of A and B are solutions of

$$\begin{cases}\frac{x^2}{a^2}-\frac{y^2}{b^2}=1\\ y=k(x-c)\end{cases} \text{ or } \begin{cases}\frac{(X+c)^2}{a^2}-\frac{k^2X^2}{b^2}=1\\ y=kX, X=x-c.\end{cases}$$

The equation $(b^2-a^2k^2)X^2+2b^2cX+b^4=0$ have two solutions X_1,X_2 with $X_1X_2=\frac{b^4}{b^2-a^2k^2}$.

So $F_1A.F_1B = \dfrac{b^4(1+k^2)}{|b^2-a^2k^2|} = \dfrac{b^2}{a^2}.\dfrac{a^2b^2(1+k^2)}{|b^2-a^2k^2|}.$

With $M\Big(\dfrac{ab}{\sqrt{|b^2-a^2k^2|}}, \dfrac{abk}{\sqrt{|b^2-a^2k^2|}}\Big) \in (H)$ we have $\dfrac{\overline{F_1A}.\overline{F_1B}}{OM^2} = -\dfrac{b^2}{a^2}.$

(2) Similary, from proposition 2.6 we have the proof. □

Exercise 2.10. Constructing the power line of two hypebols.

3. SOME IDENTITIES FOR THE CONIC SECTIONS

We proceed now to establish the fundamental identities for conic sections.

Proposition 3.1. Let A_1, A_2, A_3, A_4 be the points belong to parabola $(P): y = ax^2$ with coordinates (x_1, ax_1^2), (x_2, ax_2^2), (x_3, ax_3^2) and (x_4, ax_4^2), respectivelly, where $x_1 < x_2 < x_3 < x_4$, with 6 following points

$$M_{12}(a(x_2+x_1),1),\ \ M_{23}(a(x_2+x_3),1),\ \ M_{34}(a(x_3+x_4),1),$$

$$M_{41}(a(x_4+x_1),1),\ \ M_{13}(a(x_1+x_3),1),\ \ M_{24}(a(x_2+x_4),1)$$

of the line $d: y = 1$, we have the following identities:

(1) $\dfrac{A_1A_2}{OM_{12}} + \dfrac{A_2A_3}{OM_{23}} + \dfrac{A_3A_4}{OM_{34}} = \dfrac{A_4A_1}{OM_{41}}.$

(2) $\dfrac{A_1A_2}{OM_{12}}.\dfrac{A_3A_4}{OM_{34}} + \dfrac{A_4A_1}{OM_{41}}.\dfrac{A_2A_3}{OM_{23}} = \dfrac{A_1A_3}{OM_{13}}.\dfrac{A_2A_4}{OM_{24}}.$

Proof: (1) Direct computation shows that the relation

$$A_1A_2^2 = (x_2-x_1)^2[1+a^2(x_2+x_1)^2].$$

Then $\dfrac{A_1A_2}{OM_{12}} = x_2 - x_1.$ By an argument similar, we have 6 following relations:

$$\frac{A_1A_2}{OM_{12}} = x_2 - x_1,\ \ \frac{A_2A_3}{OM_{23}} = x_3 - x_2,\ \ \frac{A_3A_4}{OM_{34}} = x_4 - x_3,$$

$$\frac{A_4A_1}{OM_{41}} = x_4 - x_1,\ \ \frac{A_1A_3}{OM_{13}} = x_3 - x_1,\ \ \frac{A_2A_4}{OM_{24}} = x_4 - x_2.$$

Hence $\frac{A_1A_2}{OM_{12}}+\frac{A_2A_3}{OM_{23}}+\frac{A_3A_4}{OM_{34}}=x_2-x_1+x_3-x_2+x_4-x_3=\frac{A_4A_1}{OM_{41}}.$

(2) We have

$$\frac{A_1A_2}{OM_{12}}.\frac{A_3A_4}{OM_{34}}+\frac{A_4A_1}{OM_{41}}.\frac{A_2A_3}{OM_{23}}=(x_2-x_1)(x_4-x_3)+(x_4-x_1)(x_3-x_2)$$
$$=(x_4-x_2)(x_3-x_1)=\frac{A_1A_3}{OM_{13}}.\frac{A_2A_4}{OM_{24}}.$$
∎

Proposition 3.2. Let $A_1,A_2,\ldots,\ A_n,M$ be $n+1$ points belong to parabola $(P):y=ax^2$ with coordinates $A_i((x_i,ax_i^2))$ and $M(x_0,ax_0^2)$, respectivelly, where $x_1<x_2<x_3<x_4<\cdots<x_n<x_0$. With the points $I_{i(i+1)}\big(a(x_i+x_{i+1}),1\big)$, $J_i\big(a(x_0+x_i),1\big)$ belong to the line $y=1$, where $n+1\equiv 1$ and $i=1,2,\ldots,n$, we have the following identities:

(1) $\frac{A_1A_2}{OI_{12}}+\frac{A_2A_3}{OI_{23}}+\cdots+\frac{A_{n-1}A_n}{OI_{(n-1)n}}=\frac{A_nA_1}{OI_{n1}}.$

(2) $\dfrac{\frac{A_1A_2}{OI_1}}{\frac{MA_1}{OJ_1}.\frac{MA_2}{OJ_2}}+\dfrac{\frac{A_2A_3}{OI_2}}{\frac{MA_2}{OJ_2}.\frac{MA_3}{OJ_3}}+\cdots+\dfrac{\frac{A_{n-1}A_n}{OI_{n-1}}}{\frac{MA_{n-1}}{OJ_{n-1}}.\frac{MA_n}{OJ_n}}=\dfrac{\frac{A_nA_1}{OI_n}}{\frac{MA_n}{OJ_n}.\frac{MA_1}{OJ_1}}.$

Proof: (1) Arguing as in above proof, we get $\frac{A_iA_{i+1}}{OI_{i(i+1)}}=x_{i+1}-x_i$ for $i=1,\ldots,n-1$ and $\frac{A_nA_1}{OI_{n1}}=x_n-x_1$. By the computation, it is easy to verify that

$$\frac{A_1A_2}{OI_{12}}+\frac{A_2A_3}{OI_{23}}+\cdots+\frac{A_{n-1}A_n}{OI_{(n-1)n}}=x_n-x_1=\frac{A_nA_1}{OI_{n1}}.$$

(2) There is $\frac{MA_i}{OJ_i}=x_0-x_i$ for $i=1,2,\ldots,n$. By an easy computation it follows that the ratio:

$$\dfrac{\frac{A_iA_{i+1}}{OI_{i(i+1)}}}{\frac{MA_i}{OJ_i}.\frac{MA_{i+1}}{OJ_{i+1}}}=\frac{x_{i+1}-x_i}{(x_0-x_i)(x_0-x_{i+1})}=\frac{1}{x_0-x_{i+1}}-\frac{1}{x_0-x_i}.$$

Hence $\dfrac{\frac{A_1A_2}{OI_1}}{\frac{MA_1}{OJ_1}.\frac{MA_2}{OJ_2}}+\dfrac{\frac{A_2A_3}{OI_2}}{\frac{MA_2}{OJ_2}.\frac{MA_3}{OJ_3}}+\cdots+\dfrac{\frac{A_{n-1}A_n}{OI_{n-1}}}{\frac{MA_{n-1}}{OJ_{n-1}}.\frac{MA_n}{OJ_n}}=\frac{1}{x_0-x_n}-\frac{1}{x_0-x_1}=\dfrac{\frac{A_nA_1}{OI_n}}{\frac{MA_n}{OJ_n}.\frac{MA_1}{OJ_1}}.$ ∎

Definition 3.3. Let a and b be an arbitrary pair of real numbers such that $ab > 0$. A transformation under which any point $M(x, y)$ shifts to $L(ax, by)$ is called the *transformation* N_{ab}.

Clearly, under the transformation inverse N_{ab}^{-1}, any point (x, y) is sent into the point $\left(\frac{x}{a}, \frac{y}{b}\right)$.

Proposition 3.4. Let A, B, C, D be 4 points with the coordinates $(a\cos t_1, b\sin t_1)$, $(a\cos t_2, b\sin t_2)$, $(a\cos t_3, b\sin t_3)$, $(a\cos t_4, b\sin t_4)$, where $0 < t_1 < t_2 < t_3 < t_4 < 2\pi$, belong to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$. With 12 points $I_{ij}\left(a\tan\frac{t_j + t_i}{2}, b\right)$ and $M_{ij}\left(a\sin\frac{t_j + t_i}{2}, b\cos\frac{t_j + t_i}{2}\right)$, where $i, j = 1,2,3,4,\ \ i < j$, and choosing properly $u, v, t \in \{1, -1\}$ we have the following identities

(1) $\frac{AB}{OI_{12}} + u\frac{BC}{OI_{23}} + v\frac{CD}{OI_{34}} + t\frac{DA}{OI_{41}} = 0.$

(2) $\frac{AB}{OI_{12}}.\frac{DC}{OI_{34}} + u\frac{BC}{OI_{23}}.\frac{DA}{OI_{14}} + v\frac{AC}{OI_{13}}.\frac{DB}{OI_{24}} = 0.$

(3) $\frac{AB}{OM_{12}}\frac{CD}{OM_{34}} + u\frac{DA}{OM_{41}}\frac{CB}{OM_{23}} + v\frac{AC}{OM_{13}}\frac{BD}{OM_{24}} = 0.$

Proof: (1) Suppose that $A(a\cos t_1, b\sin t_1), B(a\cos t_2, b\sin t_2)$.

Then $AB = 2\sin\frac{t_2 - t_1}{2}\sqrt{a^2\sin^2\frac{t_2 + t_1}{2} + b^2\cos^2\frac{t_2 + t_1}{2}}$ or

$$\pm\frac{AB}{\sqrt{a^2\tan^2\frac{t_2 + t_1}{2} + b^2}} = \sin t_2 - \sin t_1.$$

Thus, $\pm\frac{AB}{OI_{12}} = \sin t_2 - \sin t_1$. Upon simple computation, we get

$$\frac{AB}{OI_{12}} \pm \frac{BC}{OI_{23}} \pm \frac{CD}{OI_{34}} = \sin t_4 - \sin t_1 = \pm\frac{DA}{OI_{41}}.$$

Then we obtain $\frac{AB}{OI_{12}} \pm \frac{BC}{OI_{23}} \pm \frac{CD}{OI_{34}} \pm \frac{DA}{OI_{41}} = 0.$

(2) Since $\pm\frac{AB}{OI_{12}} = \sin t_2 - \sin t_1$ and $\pm\frac{DC}{OI_{34}} = \sin t_4 - \sin t_3$ we get

$$\pm\frac{AB}{OI_{12}}.\frac{DC}{OI_{34}} = (\sin t_2 - \sin t_1)(\sin t_4 - \sin t_3).$$

Similar, there are the relations $\pm\frac{BC}{OI_{23}}.\frac{DA}{OI_{14}} = (\sin t_3 - \sin t_2)(\sin t_4 - \sin t_1)$ and $\pm\frac{AC}{OI_{13}}.\frac{DB}{OI_{24}} = (\sin t_3 - \sin t_1)(\sin t_4 - \sin t_2)$. Hence, there is the following relation

$$\frac{AB}{OI_{12}}.\frac{DC}{OI_{34}} \pm \frac{BC}{OI_{23}}.\frac{DA}{OI_{14}} \pm \frac{AC}{OI_{13}}.\frac{DB}{OI_{24}} = 0.$$

(3) follows from (2).

Lemma 3.5. Given the convex polygon $A_1A_2\ldots A_nM$ and it's circumcircle.

We have identity $\frac{A_1A_2}{MA_1.MA_2} + \frac{A_2A_3}{MA_2.MA_3} + \cdots + \frac{A_{n-1}A_n}{MA_{n-1}.MA_n} = \frac{A_nA_1}{MA_n.MA_1}$. In case $n = 3$, we get Ptolemy identity.

Proof: Assuming the circumcircle of a polygon $A_1A_2\ldots A_nM$ have radius $R = 1$. Coodinates of $A_1, A_2, \ldots, A_n, M$ are $z_1, z_2, \ldots, z_n, z$ respectivelly, where $z_k = \cos u_k + i\sin u_k$, $k = 1, 2, \ldots, n,$ and $z = \cos u + i\sin u (0 < u < u_1 < u_2 < \ldots < u_n < 2\pi)$.

We have $\frac{z_1 - z_n}{(z - z_1)(z - z_n)} = \frac{z_1 - z_2}{(z - z_1)(z - z_2)} + \frac{z_2 - z_3}{(z - z_2)(z - z_3)} + \cdots + \frac{z_{n-1} - z_n}{(z - z_{n-1})(z - z_n)}$

and
$$\frac{z_1 - z_2}{(z - z_1)(z - z_2)} = \frac{i2\sin\frac{u_2 - u_1}{2}e^{-iu}}{4\sin\frac{u_1 - u}{2}\sin\frac{u_2 - u}{2}} = \frac{iA_1A_2e^{-iu}}{MA_1.MA_2}$$

$$\frac{z_2 - z_3}{(z - z_2)(z - z_3)} = \frac{i2\sin\frac{u_3 - u_2}{2}e^{-iu}}{4\sin\frac{u_2 - u}{2}\sin\frac{u_3 - u}{2}} = \frac{iA_2A_3e^{-iu}}{MA_2.MA_3}$$

$$\cdots = \cdots$$

$$\frac{z_{n-1} - z_n}{(z - z_{n-1})(z - z_n)} = \frac{i2\sin\frac{u_n - u_{n-1}}{2}e^{-iu}}{4\sin\frac{u_n - u}{2}\sin\frac{u_{n-1} - u}{2}} = \frac{iA_{n-1}A_ne^{-iu}}{MA_{n-1}.MA_n}$$

$$\frac{z_1 - z_n}{(z - z_1)(z - z_n)} = \frac{i2\sin\frac{u_n - u_1}{2}e^{-iu}}{4\sin\frac{u_n - u}{2}\sin\frac{u_1 - u}{2}} = \frac{iA_nA_1e^{-iu}}{MA_n.MA_1}.$$

Hence $\frac{A_1A_2}{MA_1.MA_2} + \cdots + \frac{A_{n-1}A_n}{MA_{n-1}.MA_n} = \frac{A_nA_1}{MA_n.MA_1}$. □

Proposition 3.6. Let $A_1, A_2, \ldots, A_n, M$ be $n+1$ points belong to the ellipse $(E): \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the coordinates $(a\cos t_i, b\sin t_i)$ and $M(a\cos t, b\sin t)$, respectivelly, where $0 < t_1 < t_2 < t_3 < t_4 < \cdots < t_n < t < 2\pi$. With the points $I_i\left(a\sin\frac{t_{i+1} + t_i}{2}, b\cos\frac{t_{i+1} + t_i}{2}\right)$, $J_i\left(a\sin\frac{t + t_i}{2}, b\cos\frac{t + t_i}{2}\right)$, where $n + 1 \equiv 1, i = 1, 2, \ldots, n,$ and a proper choice $\pm$ we have the following identity

$$\frac{\frac{A_1A_2}{OI_1}}{\frac{MA_1}{OJ_1}\cdot\frac{MA_2}{OJ_2}} \pm \frac{\frac{A_2A_3}{OI_2}}{\frac{MA_2}{OJ_2}\cdot\frac{MA_3}{OJ_3}} \pm \cdots \pm \frac{\frac{A_{n-1}A_n}{OI_{n-1}}}{\frac{MA_{n-1}}{OJ_{n-1}}\cdot\frac{MA_n}{OJ_n}} \pm \frac{\frac{A_nA_1}{OI_n}}{\frac{MA_n}{OJ_n}\cdot\frac{MA_1}{OJ_1}} = 0.$$

Proof: Denote $B_i = N_{ab}^{-1}(A_i)$ for $i = 1,\ldots,n$. By Lemma 3.5 and $N = N_{ab}^{-1}(M)$ we have the identity $\frac{B_1B_2}{NB_1.NB_2} \pm \frac{B_2B_3}{NB_2.NB_3} \pm \cdots \pm \frac{B_{n-1}B_n}{NB_{n-1}.NB_n} \pm \frac{B_nB_1}{NB_n.NB_1} = 0.$ Because $\frac{A_1A_2}{OI_1} = B_1B_2, \ \ldots,$ $\frac{A_{n-1}A_n}{OI_{n-1}} = B_{n-1}B_n, \ \frac{A_nA_1}{OI_n} = B_nB_1$ and $\frac{MA_1}{OJ_1} = NB_1, \frac{MA_2}{OJ_2} = NB_2, \ldots, \frac{MA_n}{OJ_n} = NB_n$ we get the identity

$$\frac{\frac{A_1A_2}{OI_1}}{\frac{MA_1}{OJ_1}\cdot\frac{MA_2}{OJ_2}} \pm \frac{\frac{A_2A_3}{OI_2}}{\frac{MA_2}{OJ_2}\cdot\frac{MA_3}{OJ_3}} \pm \cdots \pm \frac{\frac{A_{n-1}A_n}{OI_{n-1}}}{\frac{MA_{n-1}}{OJ_{n-1}}\cdot\frac{MA_n}{OJ_n}} \pm \frac{\frac{A_nA_1}{OI_n}}{\frac{MA_n}{OJ_n}\cdot\frac{MA_1}{OJ_1}} = 0.$$ □

Proposition 3.7. Let A, B, C, D be 4 points belong to hypebol

$$(H): \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

with the coordinates

$$A\Big(\frac{a + ab^2t^2}{1-b^2t^2}, \frac{2b^2t}{1-b^2t^2}\Big),\ B\Big(\frac{a + ab^2u^2}{1-b^2u^2}, \frac{2b^2u}{1-b^2u^2}\Big),\ C\Big(\frac{a + ab^2v^2}{1-b^2v^2}, \frac{2b^2v}{1-b^2v^2}\Big),\ D\Big(\frac{a + ab^2z^2}{1-b^2z^2}, \frac{2b^2z}{1-b^2z^2}\Big)$$

satisfies $\frac{1}{1-b^2t^2} > \frac{1}{1-b^2u^2} > \frac{1}{1-b^2v^2} > \frac{1}{1-b^2z^2}$. With the pints

$$I_{ab}\Big(a, \frac{1+b^2tu}{t+u}\Big),\ I_{ac}\Big(a, \frac{1+b^2tv}{t+v}\Big),\ I_{ad}\Big(a, \frac{1+b^2tz}{t+z}\Big)$$

$$I_{bc}\Big(a, \frac{1+b^2uv}{u+v}\Big),\ I_{bd}\Big(a, \frac{1+b^2uz}{u+z}\Big),\ I_{cd}\Big(a, \frac{1+b^2vz}{v+z}\Big)$$

we have

(1) $\frac{AB}{OI_{ab}} + \frac{BC}{OI_{bc}} + \frac{CD}{OI_{cd}} = \frac{AD}{OI_{ad}}.$

(2) $\frac{AB}{OI_{ab}}.\frac{CD}{OI_{cd}} + \frac{AD}{OI_{ad}}.\frac{BC}{OI_{bc}} = \frac{AC}{OI_{ac}}.\frac{BD}{OI_{bd}}.$

Proof: We have

$$AB^2 = \Big(\frac{a + ab^2t^2}{1-b^2t^2} - \frac{a + ab^2u^2}{1-b^2u^2}\Big)^2 + \Big(\frac{2b^2t}{1-b^2t^2} - \frac{2b^2u}{1-b^2u^2}\Big)^2.$$

Thus, $AB = \frac{2b^2|t^2-u^2|\sqrt{a^2 + \big(\frac{1+b^2tu}{t+u}\big)^2}}{|1-b^2t^2||1-b^2u^2|}$. By computation the relation

$$\frac{AB}{OI_{ab}} = \frac{2b^2|t^2-u^2|}{|1-b^2t^2||1-b^2u^2|} = 2\Big(\frac{1}{1-b^2t^2} - \frac{1}{1-b^2u^2}\Big)$$

$$\frac{AC}{OI_{ac}} = 2\left|\frac{1}{1-b^2t^2} - \frac{1}{1-b^2v^2}\right| = 2\left(\frac{1}{1-b^2t^2} - \frac{1}{1-b^2v^2}\right)$$
$$\frac{AD}{OI_{ad}} = 2\left|\frac{1}{1-b^2t^2} - \frac{1}{1-b^2z^2}\right| = 2\left(\frac{1}{1-b^2t^2} - \frac{1}{1-b^2z^2}\right)$$
$$\frac{BC}{OI_{bc}} = 2\left|\frac{1}{1-b^2u^2} - \frac{1}{1-b^2v^2}\right| = 2\left(\frac{1}{1-b^2u^2} - \frac{1}{1-b^2v^2}\right)$$
$$\frac{BD}{OI_{bd}} = 2\left|\frac{1}{1-b^2u^2} - \frac{1}{1-b^2z^2}\right| = 2\left(\frac{1}{1-b^2u^2} - \frac{1}{1-b^2z^2}\right)$$
$$\frac{CD}{OI_{cd}} = 2\left|\frac{1}{1-b^2v^2} - \frac{1}{1-b^2z^2}\right| = 2\left(\frac{1}{1-b^2v^2} - \frac{1}{1-b^2z^2}\right).$$

Hence, we obtain $\frac{AB}{OI_{ab}}.\frac{CD}{OI_{cd}} + \frac{AD}{OI_{ad}}.\frac{BC}{OI_{bc}} = \frac{AC}{OI_{ac}}.\frac{BD}{OI_{bd}}$ and (2). We have $\frac{AB}{OI_{ab}} = 2\left(\frac{1}{1-b^2t^2} - \frac{1}{1-b^2u^2}\right)$. The anothers relations are proved in an analogous fashion. Then, we have the identity $\frac{AB}{OI_{ab}} + \frac{BC}{OI_{bc}} + \frac{CD}{OI_{cd}} = \frac{AD}{OI_{ad}}$ and (1). □

Proposition 3.8. Let $A_1, A_2, \ldots, A_n, A_{n+1}$ be $n+1$ points belong to hypebol $(H): \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with the coordinates $A_i\left(\frac{a + ab^2t_i^2}{1-b^2t_i^2}, \frac{2b^2t_i}{1-b^2t_i^2}\right)$, $i = 1, 2, \ldots, n+1$, and $M\left(\frac{a+ab^2t^2}{1-b^2t^2}\right)$ satisfy the conditions $\frac{1}{1-b^2t_i^2} > \frac{1}{1-b^2t_{i+1}^2}$, where $i = 1, 2, \ldots, n$, and the points $I_{r,s}\left(a, \frac{1+b^2t_rt_s}{t_r+t_s}\right)$, $r, s = 1, 2, \ldots, n+1$. Then, we have the following identities:

(1) $\sum_{i=1}^{n} \frac{A_iA_{i+1}}{OI_{i,i+1}} = \frac{A_1A_{n+1}}{OI_{1,n+1}}$.

(2) $\sum_{i=1}^{n} \frac{\frac{A_iA_{i+1}}{OI_{i,i+1}}}{\frac{A_iA_{n+1}}{OI_{i,n+1}}\frac{A_{i+1}A_{n+1}}{OI_{i+1,n+1}}} = \frac{\frac{A_nA_1}{OI_{n,1}}}{\frac{A_nA_{n+1}}{OI_{n,n+1}}\frac{A_1A_{n+1}}{OI_{1,n+1}}}$.

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