ORIGINAL PAPER

ON INTEGRALS INVOLVING THE GENERALIZED I-FUNCTION

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Abstract. *This paper deals with the study of the integral representations involving the product of the Aleph function with exponential function, Gauss hypergeometrc function and Fox's H-function. The results derived in this paper are basic in nature and include a number of known and new results as special cases.*

Keywords: *Aleph function, Mellin-Barnes type integrals, H -function, I-function, exponential function, hypergeometric function.*

2010 Mathematics Subject Classifications: *33C60, 33C20, 44A20.*

1. INTRODUCTION AND PRELIMINARIES

In the past century, many authors have generalized H-function. In a recent paper, Saxena et al. [17] have introduced a generalization of Saxena's I-function [30]. This is also a generalization of Fox's H-function. Saxena and Pogany [18] have studied fractional integral formulae for the Aleph function. Sudland et al. [8] studied the generalized fractional driftless Fokker-Planck equation with power law coefficient. As a result a special function was found, which is a particular case of the Aleph function. In order to unify and extend the results for the convergent Mathieu–type a-series and its alternative form series and alternating Mathieu-type a-series whose terms contain the familiar transcendental functions, such as Gauss hypergeometric function, generalized hypergeometric function, the Fox-Wright function, the Meijer's G-function, Fox's H-function, published in a series of papers by Pogany[20 - 22], Pogany et al.[23 - 26], Srivastava and Tomovski[5, 32], Tomovski and Tuan [35], the authors introduced the Mathieu-type a-series and its alternative variant, whose terms contain an I-function. Inequalities for Mathieu-type series are discussed by Cerone [10], Pogany and Tomovski [28], Srivastava and Tomovski [5], Tomovski and Hilfer [33] and Tomovski and Pogany [34]. The results obtained by the authors serve as the key formulas for numerous potentially useful special functions of Science, Engineering and Technology scattered in the literature.

In 1961, Charles Fox [3] introduced a function which is more general in than the Meijer's G-function and this function is well known in the literature of special functions as Fox's H-function. The function is defined and presented by means of the following Mellin-Barnes type contour integral:

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$$H(z) = H_{p,q}^{m,n}[z] = H_{p,q}^{m,n}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,p} \\ (b_j,\beta_j)_{1,q} \end{matrix} \right] = \frac{1}{2\pi i}\int_L \theta(s) z^s ds, \tag{1}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\prod_{j=m+1}^{q}\Gamma(1 - b_j + \beta_j s)\prod_{j=n+1}^{p}\Gamma(a_j - \alpha_j s)}. \tag{2}$$

An account of the convergence conditions for this integral can be found in the paper by Fox [3].

The I-function which is more general than the Fox's H-function, defined by Saxena [17], by means of the following Mellin-Barnes type contour integral:

$$I[z] = I_{p_i,q_i:r}^{m,n}[z] = I_{p_i,q_i:r}^{m,n}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,n};(a_{ji},\alpha_{ji})_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};(b_{ji},\beta_{ji})_{m+1,q_i} \end{matrix} \right] = \frac{1}{2\pi i}\int_L \theta(s) z^s ds, \tag{3}$$

where

$$\theta(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j - \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j + \alpha_j s)}{\sum_{i=1}^{r}\left\{\prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} + \beta_{ji} s)\prod_{j=n+1}^{p_i}\Gamma(a_{ji} - \alpha_{ji} s)\right\}}. \tag{4}$$

For details regarding existence conditions and variour parameter restrictions of I-function we may refer [17].

The Aleph function which is a generalization of the H-function and I-function, recently introduced by Saxena et al.[14], by means of the following Mellin-Barnes type contour integral:

$$\aleph[z] = \aleph_{p_i,q_i,\tau_i:r}^{m,n}[z] = \aleph_{p_i,q_i,\tau_i:r}^{m,n}\left[z \middle| \begin{matrix} (a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta i_j)]_{m+1,q_i} \end{matrix} \right] = \frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i,\tau_i:r}^{m,n}(s) z^{-s} ds, \tag{5}$$

For all $z \neq 0$, where $\omega = \sqrt{-1}$ and

$$\Omega_{p_i,q_i,\tau_i:r}^{m,n}(s) = \frac{\prod_{j=1}^{m}\Gamma(b_j + \beta_j s)\prod_{j=1}^{n}\Gamma(1 - a_j - \alpha_j s)}{\sum_{i=1}^{r}\tau_i\left\{\prod_{j=n+1}^{p_i}\Gamma(a_{ji} + \alpha_{ji} s)\prod_{j=m+1}^{q_i}\Gamma(1 - b_{ji} - \beta_{ji} s)\right\}}. \tag{6}$$

where $\tau_i > 0$.

An account of the convergence conditions for the above integral can be found in the paper by Saxena and Pogány ([14, 15]).

In 1812, Carl Friedrich Gauss introduced the function [4]

$${}_2F_1[\alpha,\beta;\gamma;x] = \sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k x^k}{(\gamma)_k k!} \tag{7}$$

is called Gauss's hypergeometric function and $(\alpha)_k$ is the Pochhammer symbol given by
$(\alpha)_n = \alpha(\alpha+1)...(\alpha+(n-1))$ with $(\alpha)_0 = 1, n \in N.$

From [4], we have

$$\sum_{n=0}^{\infty}\sum_{k=0}^{\infty} A(k,n) = \sum_{n=0}^{\infty}\sum_{k=0}^{n} A(k,n-k) \tag{8}$$

2. MAIN RESULTS

In this section we will evaluate the certain integrals involving the product of the Aleph function with exponential function, Gauss hypergeometrc function and Fox's H-function. The results are presented in the form of theorems stated below:

Theorem 2.1

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-zx}\ {}_2F_1(\alpha,\beta,;\gamma; a\, x^{\zeta}(t-x)^{\eta})$$

$$\times \aleph^{m,n}_{p_i,q_i,\tau_i:r}\left[y\, x^{\mu}(t-x)^{\upsilon} \left| \begin{matrix} (a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right. \right] dx$$

$$= e^{-zt}\, t^{\rho+\sigma-1} \sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k) \frac{z^{u-k}}{(u-k)!} t^{(\varsigma+\eta-1)k+u}$$

$$\times \aleph^{m,n+2}_{p_i+2,q_i+1,\tau_i:r}\left[y\, t^{\mu+\upsilon} \left| \begin{matrix} (1-\rho-\varsigma k,\mu),(1-\sigma-(\eta-1)k-u,\upsilon),(a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta i_j)]_{m+1,q_i},(1-\rho-\sigma-(\varsigma+\eta-1)k-u,\mu+\upsilon) \end{matrix} \right. \right], \tag{9}$$

where

$$f(k) = \frac{(\alpha)_k(\beta)_k\, a^k}{(\gamma)_k k!} \tag{10}$$

provided (i) $\tau_i > 0, \mu, \upsilon \geq 0,$ (not both zero simultaneously)

(ii) ς and η are non-negative integers such that $\varsigma + \eta \geq 1$

(iii) $A_i > 0, B_i < 0; \left|\arg y\right| < \frac{1}{2} A_i \pi,$ for all $i = 1,2,...,r;$ where

$$A_i = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ii} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ii},\ B_i = \tfrac{1}{2}(p_i - q_i) + \sum_{j=1}^{q_i} b_{jij} - \sum_{j=1}^{p_i} a_{ii}$$

(iv) $\mathrm{Re}(\rho) + \mu \min_{1\leq j\leq m}[\mathrm{Re}(b_j/\beta_j] > 0, \mathrm{Re}(\sigma) + \upsilon \min_{1\leq j\leq m}[\mathrm{Re}(b_j/\beta_j] > 0.$

Proof: Taking LHS

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-zx}\ {}_2F_1(\alpha,\beta,;\gamma; a\, x^{\zeta}(t-x)^{\eta}) \times \aleph^{m,n}_{p_i,q_i,\tau_i:r}\left[y\, x^{\mu}(t-x)^{\upsilon} \left| \begin{matrix} (a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right. \right] dx$$

It can be written as

$$e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}e^{(t-x)z}\ {}_2F_1(\alpha,\beta,;\gamma;a\,x^{\varsigma}(t-x)^{\eta})$$
$$\times\aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[y\,x^{\mu}(t-x)^{\upsilon}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i}\end{matrix}\right.\right]dx$$

By writing exponential and gauss hypergeometric functions in their series form and using the definition of Aleph function (5), we get

$$=e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k\,a^k\,x^{\varsigma k}(t-x)^{\eta k+u}\,z^u}{(\gamma)_k k!\,u!}$$
$$\times\frac{1}{2\pi\omega}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\varsigma)\,y^{\varsigma}x^{\mu\varsigma}(t-x)^{\upsilon\varsigma}\,d\varsigma\,dx \tag{11}$$

Using (8) in (11), we have

$$=e^{-zt}\int_0^t x^{\rho-1}(t-x)^{\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k\,a^k\,x^{\varsigma k}(t-x)^{\eta k+u-k}\,z^{u-k}}{(\gamma)_k k!\,(u-k)!}$$
$$\times\frac{1}{2\pi\omega}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\varsigma)\,y^{\varsigma}x^{\mu\varsigma}(t-x)^{\upsilon\varsigma}\,d\varsigma\,dx$$

On interchanging the order of integration and summation, we arrive at

$$=e^{-zt}\sum_{u=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k\,z^{u-k}}{(\gamma)_k(u-k)!}\frac{1}{2\pi\omega}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\varsigma)\,y^{\varsigma}$$
$$\times\left\{\int_0^t x^{\rho+\varsigma k+\mu\varsigma-1}(t-x)^{\sigma+(\eta-1)k+u+\upsilon\varsigma-1}\,dx\right\}d\varsigma$$

Setting $x = ts$ in the inner integral, the above expression becomes

$$=e^{-zt}\,t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k\,z^{u-k}}{(\gamma)_k(u-k)!}t^{(\varsigma+\eta-1)k+u}\frac{1}{2\pi\omega}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\varsigma)\,t^{(\mu+\upsilon)\varsigma}\,y^{\varsigma}$$
$$\times\left\{\int_0^1 s^{\rho+\varsigma k+\mu\varsigma-1}(1-s)^{\sigma+(\eta-1)k+u+\upsilon\varsigma-1}\,ds\right\}d\varsigma$$

By the definition of beta function, we get

$$=e^{-zt}\,t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}\frac{(\alpha)_k(\beta)_k\,z^{u-k}}{(\gamma)_k(u-k)!}t^{(\varsigma+\eta-1)k+u}$$
$$\times\frac{1}{2\pi\omega}\int_L\Omega^{m,n}_{p_i,q_i,\tau_i;r}(\varsigma)\frac{\Gamma(\rho+\varsigma k+\mu\varsigma)\Gamma(\sigma+(\eta-1)k+u+\upsilon\varsigma)}{\Gamma(\rho+\sigma+(\varsigma+\eta-1)k+u+(\mu+\upsilon)\varsigma)}t^{(\mu+\upsilon)\varsigma}\,y^{\varsigma} \tag{12}$$

Finally, using the definition (5), we get the desired result.

Theorem 2.2

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-zx} \;{}_2F_1(\alpha,\beta,;\gamma;a\,x^{\zeta}(t-x)^{\eta})$$
$$\times \aleph^{m,n}_{p_i,q_i,\tau_i:r}\left[y\,x^{-\mu}(t-x)^{-\upsilon} \middle| \begin{matrix} (a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix}\right] dx$$
$$= e^{-zt}\, t^{\rho+\sigma-1} \sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!} t^{(\varsigma+\eta-1)k+u}$$
$$\times \aleph^{m+2,n}_{p_i+1,q_i+2,\tau_i:r}\left[y\,t^{-\mu-\upsilon} \middle| \begin{matrix} (a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i},(\rho+\sigma+(\varsigma+\eta-1)k+u,\mu+\upsilon) \\ (\rho+\varsigma k,\mu),(\sigma+(\eta-1)k+u,\upsilon),(b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix}\right], \qquad (13)$$

provided (i) $\tau_i > 0, \mu,\upsilon \geq 0,$ (not both zero simultaneously)

(ii) ς and η are non-negative integers such that $\varsigma+\eta \geq 1$

(iii) $A_i > 0, B_i < 0; |\arg y| < \frac{1}{2}A_i\pi,$ for all $i = 1,2,...,r;$ where

$$A_i = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ii} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ii},\; B_i = \tfrac{1}{2}(p_i-q_i) + \sum_{j=1}^{q_i} b_{jij} - \sum_{j=1}^{p_i} a_{ii}$$

(iv) $\text{Re}(\rho) - \mu \max_{1\leq j\leq n}[\text{Re}(a_j-1)/\alpha_j] > 0, \text{Re}(\sigma) - \upsilon \max_{1\leq j\leq n}[\text{Re}(a_j-1)/\alpha_j] > 0.$

and $f(k)$ is given by (10).

Theorem 2.3

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-zx} \;{}_2F_1(\alpha,\beta,;\gamma;a\,x^{\zeta}(t-x)^{\eta})$$
$$\times \aleph^{m,n}_{p_i,q_i,\tau_i:r}\left[y\,x^{\mu}(t-x)^{-\upsilon} \middle| \begin{matrix} (a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix}\right] dx$$
$$= e^{-zt}\, t^{\rho+\sigma-1} \sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!} t^{(\varsigma+\eta-1)k+u}$$
$$\times \aleph^{m+1,n+1}_{p_i+1,q_i+2,\tau_i:r}\left[y\,t^{\mu-\upsilon} \middle| \begin{matrix} (1-\rho-\varsigma k+u,\mu),(a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (\sigma+(\eta-1)k+u,\upsilon),(b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta i_j)]_{m+1,q_i},(1-\rho-\sigma-(\varsigma+\eta-1)k-u,\mu-\upsilon) \end{matrix}\right], \qquad (14)$$

provided (i) $\tau_i > 0, \mu,\upsilon \geq 0,$ (not both zero simultaneously)

(ii) ς and η are non-negative integers such that $\varsigma+\eta \geq 1$

(iii) $A_i > 0, B_i < 0; |\arg y| < \frac{1}{2}A_i\pi,$ for all $i = 1,2,...,r;$ where

$$A_i = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ii} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ii},\; B_i = \tfrac{1}{2}(p_i-q_i) + \sum_{j=1}^{q_i} b_{jij} - \sum_{j=1}^{p_i} a_{ii}$$

(iv) $\mu > 0, \upsilon \geq 0,$ such that $\mu - \upsilon \geq 0,$

$$\text{Re}(\rho) + \mu \min_{1\leq j\leq m}[\text{Re}(b_j/\beta_{j]} > 0] > 0, \text{Re}(\sigma) - \upsilon \max_{1\leq j\leq n}[\text{Re}(a_j-1)/\alpha_j] > 0.$$

and $f(k)$ is given by (10).

Theorem 2.4

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-zx} \,{}_2F_1(\alpha,\beta,;\gamma; a\, x^{\zeta}(t-x)^{\eta})$$
$$\times \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[y\, x^{\mu}(t-x)^{-\upsilon} \left| \begin{matrix} (a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right. \right] dx$$
$$= e^{-zt}\, t^{\rho+\sigma-1} \sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!} t^{(\varsigma+\eta-1)k+u}$$
$$\times \aleph^{m+1,n+1}_{p_i+2,q_i+1,\tau_i;r}\left[y\, t^{\mu-\upsilon} \left| \begin{matrix} (1-\rho-\varsigma k,\mu),(a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i},(\rho+\sigma+(\varsigma+\eta-1)k+u,\upsilon-\mu) \\ (\sigma+(\eta-1)k+u,\upsilon),(b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right. \right], \quad (15)$$

provided (i) $\tau_i > 0, \mu, \upsilon \geq 0$, (not both zero simultaneously)

(ii) ς and η are non-negative integers such that $\varsigma + \eta \geq 1$

(iii) $A_i > 0, B_i < 0; |\arg y| < \frac{1}{2} A_i \pi$, for all $i = 1,2,...,r$; where

$$A_i = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ii} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ii},\; B_i = \frac{1}{2}(p_i - q_i) + \sum_{j=1}^{q_i} b_{jij} - \sum_{j=1}^{p_i} a_{ii}$$

(iv) $\mu \geq 0, \upsilon > 0$, such that $\upsilon - \mu \geq 0$,

$$\mathrm{Re}(\rho) - \mu \max_{1\leq j\leq n}[\mathrm{Re}(a_j - 1)/\alpha_j > 0] > 0, \mathrm{Re}(\sigma) + \upsilon \min_{1\leq j\leq m}[\mathrm{Re}(b_j/\beta_j)] > 0.$$

and $f(k)$ is given by (10).

Theorem 2.5

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-zx} \,{}_2F_1(\alpha,\beta,;\gamma; a\, x^{\zeta}(t-x)^{\eta})$$
$$\times \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[y\, x^{-\mu}(t-x)^{\upsilon} \left| \begin{matrix} (a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i} \\ (b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right. \right] dx$$
$$= e^{-zt}\, t^{\rho+\sigma-1} \sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!} t^{(\varsigma+\eta-1)k+u}$$
$$\times \aleph^{m+1,n+1}_{p_i+2,q_i+1,\tau_i;r}\left[y\, t^{-\mu+\upsilon} \left| \begin{matrix} (1-\sigma-(\varsigma-1)k-u,\upsilon),(a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i},(\rho+\sigma+(\varsigma+\eta-1)k+u,\mu-\upsilon) \\ (\rho+\varsigma k,\mu),(b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i} \end{matrix} \right. \right], \quad (16)$$

provided (i) $\tau_i > 0, \mu, \upsilon \geq 0$, (not both zero simultaneously)

(ii) ς and η are non-negative integers such that $\varsigma + \eta \geq 1$

(iii) $A_i > 0, B_i < 0; |\arg y| < \frac{1}{2} A_i \pi$, for all $i = 1,2,...,r$; where

$$A_i = \sum_{j=1}^{n}\alpha_j - \sum_{j=n+1}^{p_i}\alpha_{ii} + \sum_{j=1}^{m}\beta_j - \sum_{j=m+1}^{q_i}\beta_{ii},\; B_i = \frac{1}{2}(p_i - q_i) + \sum_{j=1}^{q_i} b_{jij} - \sum_{j=1}^{p_i} a_{ii}$$

(iv) $\mu > 0, \upsilon \geq 0$, such that $\mu - \upsilon \geq 0$,

$$\mathrm{Re}(\rho) + \mu \min_{1\leq j\leq m}[\mathrm{Re}(b_j/\beta_j] > 0] > 0, \mathrm{Re}(\sigma) - \upsilon \max_{1\leq j\leq n}[\mathrm{Re}(a_j - 1)/\alpha_j] > 0.$$

and $f(k)$ is given by (10).

Theorem 2.6

$$\int_0^t x^{\rho-1}(t-x)^{\sigma-1} e^{-zx}\ {}_2F_1(\alpha,\beta,;\gamma;a\,x^{\zeta}(t-x)^{\eta})$$
$$\times \aleph^{m,n}_{p_i,q_i,\tau_i;r}\left[y\,x^{-\mu}(t-x)^{\upsilon}\left|\begin{matrix}(a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta i_j)]_{m+1,q_i}\end{matrix}\right.\right]dx$$
$$=e^{-zt}\,t^{\rho+\sigma-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{z^{u-k}}{(u-k)!}t^{(\varsigma+\eta-1)k+u}$$
$$\times \aleph^{m+1,n+1}_{p_i+1,q_i+2,\tau_i;r}\left[y\,t^{-\mu+\upsilon}\left|\begin{matrix}(1-\sigma-(\eta-1)k-u,\upsilon),(a_j,\alpha_j)_{1,n};[\tau_j(a_{ji},\alpha_{ji})]_{n+1,p_i}\\(\rho+\varsigma k,\mu),(b_j,\beta_j)_{1,m};[\tau_j(b_{ji},\beta_{ji})]_{m+1,q_i},(1-\rho-\sigma-(\varsigma+\eta-1)k-u,\upsilon-\mu)\end{matrix}\right.\right], \quad (17)$$

provided (i) $\tau_i>0,\mu,\upsilon\geq 0,$ (not both zero simultaneously)

(ii) ς and η are non-negative integers such that $\varsigma+\eta\geq 1$

(iii) $A_i>0, B_i<0; |\arg y|<\frac{1}{2}A_i\pi,$ for all $i=1,2,...,r;$ where

$A_i=\sum_{j=1}^{n}\alpha_j-\sum_{j=n+1}^{p_i}\alpha_{ii}+\sum_{j=1}^{m}\beta_j-\sum_{j=m+1}^{q_i}\beta_{ii}, B_i=\frac{1}{2}(p_i-q_i)+\sum_{j=1}^{q_i}b_{jij}-\sum_{j=1}^{p_i}a_{ii}$

(iv) $\mu\geq 0,\upsilon>0,$ such that $\upsilon-\mu\geq 0,$

$\mathrm{Re}(\rho)-\mu\max_{1\le j\le n}[\mathrm{Re}(a_j-1)/\alpha_j>0]>0, \mathrm{Re}(\sigma)+\upsilon\min_{1\le j\le m}[\mathrm{Re}(b_j/\beta_j)]>0.$

and $f(k)$ is given by (10).

Theorem 2.7

$$\int_0^{\infty} x^{\eta-1}e^{ax}\ {}_2F_1(\alpha,\beta,;\gamma;a\,x^{\rho}(t-x)^{\eta})$$
$$\times \aleph^{m_1,n_1}_{p_i,q_i,\tau_i;r}\left[z\,x^{\rho}(t-x)^{\upsilon}\left|\begin{matrix}(a_j,\alpha_j)_{1,n_1};[\tau_j(a_{ji},\alpha_{ji})]_{n_1+1,p_i}\\(b_j,\beta_j)_{1,m_1};[\tau_j(b_{ji},\beta_{ji})]_{m_1+1,q_i}\end{matrix}\right.\right]\times H^{m,n}_{p,q}\left[\omega x\left|\begin{matrix}(c_j,\gamma_j)_{1,n};(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m};(d_j,\delta_{jj})_{m+1,q}\end{matrix}\right.\right]dx$$
$$=\omega^{-\eta}\sum_{u=0}^{\infty}\sum_{k=0}^{n} f(k)\frac{a^{u-k}}{(u-k)!}\omega^{-(\rho-1)k-u}$$
$$\times \aleph^{m_1+n,n_1+m}_{p_i+q,q_i+p,\tau_i;r}\left[z\,\omega^{-\sigma}\left|\begin{matrix}(a_j,\alpha_j)_{1,n_1},(1-d_j-(\eta+(\rho-1)k+u)\delta_j,\sigma\delta_j)_{1,m};\\(b_j,\beta_j)_{1,m_1},(1-c_j-(\eta+(\rho-1)k+u)\gamma_j,\sigma\gamma_j)_{1,n};\end{matrix}\right.\right.$$
$$\left.\begin{matrix}[\tau_j(a_{ji},\alpha_{ji})]_{n_1+1,p_i},(1-d_j-(\eta+(\rho-1)k+u)\delta_j,\sigma\delta_j)_{m+1,q}\\ [\tau_j(b_{ji},\beta_{ji})]_{m_1+1,q_i},(1-c_j-(\eta+(\rho-1)k+u)\gamma_j,\sigma\gamma_j)_{n+1,p}\end{matrix}\right] \quad (18)$$

provided (i) $\tau_i>0,\lambda>0,|\arg z|<\frac{1}{2}\pi\lambda$

(ii) $\lambda\geq 0,|\arg z|\leq\frac{1}{2}\pi\lambda, \mathrm{Re}(\mu+1)<0$

(iii) $\lambda_1 > 0, |\arg\omega| < \frac{1}{2}\pi\lambda_1$

(iv) $\lambda_1 \geq 0, |\arg\omega| \leq \frac{1}{2}\pi\lambda_1, \mathrm{Re}(\mu_1+1) < 0$

$\sigma > 0, -\sigma \min_{1 \leq j \leq m_1}[\mathrm{Re}(b_j/\beta_j)] - \min_{1 \leq j \leq m}[\mathrm{Re}(d_j/\delta_j)] < \mathrm{Re}(\eta) < \sigma <$
$\min_{1 \leq j \leq n_1}[\mathrm{Re}(a_j-1)/\alpha_j] + \min_{1 \leq j \leq n}[\mathrm{Re}(1-c_j)/\gamma_j)]$

and where

$$\lambda = \sum_{j=1}^{n_1}\alpha_j + \sum_{j=1}^{m_1}\beta_j - \max_{1 \leq i \leq r}\left[\sum_{j=n_1+1}^{p_i}\alpha_{ji} + \sum_{j=m_1+1}^{q_i}\beta_{ji}\right]$$

$$\mu = \sum_{j=1}^{m_1}b_j - \sum_{j=1}^{n_1}a_j - \min_{1 \leq i \leq r}\left[\sum_{j=n_1+1}^{p_i}a_{ji} - \sum_{j=m_1+1}^{q_i}b_{ji} + \frac{p_i}{2} - \frac{q_i}{2}\right]$$

$$\lambda_1 = \sum_{j=1}^{m}\delta_j - \sum_{j=1}^{n}\gamma_j - \sum_{j=m+1}^{q}\delta_j - \sum_{j=n+1}^{p}\gamma_j$$

$$\mu_1 = \frac{1}{2}(p-q) + \sum_{j=1}^{q}d_j - \sum_{j=1}^{p}c_j.$$

Proof: By using the definitions of exponential function, Gauss gypergeometric function and the Aleph function in series form in the LHS, we get

$$\int_0^\infty x^{\eta-1}\sum_{u=0}^{\infty}\frac{a^u x^u}{u!}\sum_{k=0}^{\infty}\frac{(\alpha)_k(\beta)_k a^k x^{\rho k}}{(\gamma)_k k!}\frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(\varsigma)z^{\varsigma}x^{\sigma\varsigma}$$
$$\times H_{p,q}^{m,n}\left[\omega x\left|\begin{matrix}(c_j,\gamma_j)_{1,n};(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m};(d_j,\delta_{jj})_{m+1,q}\end{matrix}\right.\right]d\varsigma dx$$
$$= \int_0^\infty x^{\eta-1}\sum_{u=0}^{\infty}\sum_{k=0}^{\infty}\frac{a^u}{u!}\frac{(\alpha)_k(\beta)_k a^k x^{\rho k+u}}{(\gamma)_k k!}\frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(\varsigma)z^{\varsigma}x^{\sigma\varsigma}$$
$$\times H_{p,q}^{m,n}\left[\omega x\left|\begin{matrix}(c_j,\gamma_j)_{1,n};(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m};(d_j,\delta_{jj})_{m+1,q}\end{matrix}\right.\right]d\varsigma dx \tag{19}$$

In view of (8), above expression reduces to

$$= \int_0^\infty x^{\eta-1}\sum_{u=0}^{\infty}\sum_{k=0}^{n}\frac{1}{(u-k)!}\frac{(\alpha)_k(\beta)_k a^{u-k} x^{\rho k+u-k}}{(\gamma)_k k!}\frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(\varsigma)z^{\varsigma}x^{\sigma\varsigma}$$
$$\times H_{p,q}^{m,n}\left[\omega x\left|\begin{matrix}(c_j,\gamma_j)_{1,n};(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m};(d_j,\delta_{jj})_{m+1,q}\end{matrix}\right.\right]d\varsigma dx$$

On interchanging the order of integration and summation, we get

$$= \sum_{u=0}^{\infty}\sum_{k=0}^{n}f(k)\frac{a^{u-k}}{(u-k)!}\frac{1}{2\pi\omega}\int_L \Omega_{p_i,q_i,\tau_i;r}^{m,n}(\varsigma)z^{\varsigma}x^{\sigma\varsigma}$$
$$\times\left\{\int_0^\infty x^{\eta+(\rho-1)k+u+\sigma\varsigma-1}\times H_{p,q}^{m,n}\left[\omega x\left|\begin{matrix}(c_j,\gamma_j)_{1,n};(c_j,\gamma_j)_{n+1,p}\\(d_j,\delta_j)_{1,m};(d_j,\delta_{jj})_{m+1,q}\end{matrix}\right.\right]dx\, d\varsigma\right\} \tag{20}$$

where $f(k)$ is given by (10).

Now by using the definition of H-function, we obtain

$$= \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{a^{u-k}}{(u-k)!} \frac{1}{2\pi\omega} \int_L \Omega_{p_i,q_i,\tau_i:r}^{m,n}(\varsigma) z^{\varsigma} \omega^{-(\eta+(\rho-1)k+u+\sigma\varsigma)}$$

$$\times \frac{\prod_{j=1}^{m} \Gamma(d_j + \delta_j(\eta + (\rho-1)k + u + \sigma\varsigma))}{\prod_{j=m+1}^{q} \Gamma(1 - d_j - \delta_j(\eta + (\rho-1)k + u + \sigma\varsigma))}$$

$$\times \frac{\prod_{j=1}^{n} \Gamma(1 - c_j - \gamma_j(\eta + (\rho-1)k + u + \sigma\varsigma))}{\prod_{j=n+1}^{p} \Gamma(c_j + \gamma_j(\eta + (\rho-1)k + u + \sigma\varsigma))} d\varsigma$$

$$= \omega^{-\eta} \sum_{u=0}^{\infty} \sum_{k=0}^{n} f(k) \frac{a^{u-k}}{(u-k)!} \omega^{-(\rho-1)k-u} \frac{1}{2\pi\omega} \int_L \Omega_{p_i,q_i,\tau_i:r}^{m,n}(\varsigma)$$

$$\times \frac{\prod_{j=1}^{m} \Gamma(d_j + \delta_j(\eta + (\rho-1)k + u) + \sigma\delta_j\varsigma)}{\prod_{j=m+1}^{q} \Gamma(1 - d_j - \delta_j(\eta + (\rho-1)k + u) - \sigma\delta_j\varsigma)}$$

$$\times \frac{\prod_{j=1}^{n} \Gamma(1 - c_j - \gamma_j(\eta + (\rho-1)k + u) + \sigma\varsigma)}{\prod_{j=n+1}^{p} \Gamma(c_j + \gamma_j(\eta + (\rho-1)k + u) + \sigma\varsigma)} z^{\varsigma} \omega^{-\sigma\varsigma} d\varsigma \qquad (21)$$

Finally, interpreting the integral by virtue of (5), we arrive at the desired result.

3. SPECIAL CASES

As Aleph function is the most generalized special function, numerous special cases with potentially useful transcendental functions such as Mittag-Leffler function, Bessel functions, Whittaker functions, hypergeometric functions, generalized hypergeometric function, Meijer's G-function, Fox-Wright function, Fox's H-function and I-function and their special cases can be deduced by assigning suitable values to the parameters. Some interesting special cases of the main results are given below:

(i) If we set $\tau_i = 1$ in theorems (2.1) to (2.7), then results given by Saha et al. [16] are recovered.

(ii) If we put $\tau_i = r = t = 1, \eta = 0$ in theorems (2.1) to (2.4), which lead to the well-known result [8].

(iii) If we take $\tau_i = r = t = 1, \eta = 0$ in theorems (2.5 and (2.6), which lead to the result [16].

(iv) If we set $\tau_i = r = t = 1, \eta = 0, a = 0$ in theorem (2.7), which reduces to the result [16].

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