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SOME REFINEMENTS OF HOLDER'S INEQUALITIES VIA ISOTONIC LINEAR FUNCTIONALS

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Abstract: *In this paper are given some generalizations of Holder's inequalities for isotonic linear functionals using several generalizations of Young's inequality, Kittaneh-Manasrah's inequality and the difference-type reverse inequality. Then, as applications, these inequalities will be rewritten for several important particular cases of isotonic linear functionals.*

Keywords: *Young's inequality, Holder's inequality, Kittaneh-Manasrah's inequality.*

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1. INTRODUCTION

It is necessary to recall below an inequality given in Theorem 2.1 in the paper of N. Minculete, see [4].

Theorem 1. [4] For $a,\ b \geq 1$ and $\lambda \in (0,1)$, we have

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 + A(\lambda)\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq$$

$$\leq (1-r)\left(\sqrt{a}-\sqrt{b}\right)^2 + B(\lambda)\log^2\left(\frac{a}{b}\right)$$

where $r = \min\{\lambda,\ 1-\lambda\}$, $A(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{r}{4}$ and $B(\lambda) = \frac{\lambda(1-\lambda)}{2} - \frac{1-r}{4}$.

In the case when $0 < a,b \leq 1$ and $\lambda \in (0,1)$ it is known like an application the following result:

Application 2. [4] For $0 < a,b \leq 1$ and $\lambda \in (0,1)$, we have

$$r\left(\sqrt{a}-\sqrt{b}\right)^2 + A(\lambda)ab.\log^2\left(\frac{a}{b}\right) \leq \lambda a + (1-\lambda)b - a^\lambda b^{1-\lambda} \leq$$

$$\leq (1-r)\left(\sqrt{a}-\sqrt{b}\right)^2 + B(\lambda)ab.\log^2\left(\frac{a}{b}\right)$$

where $r = \min\{\lambda,\ 1-\lambda\}$ and $A(\lambda), B(\lambda)$ are given before in Theorem 1.

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The following inequality, given in [3], is a reverse of Young's inequality, the difference-type reverse inequality. This result will be also used below in the proof of a theorem of this paper.

Corollary 1. [3] For $a,b>0$ and $\lambda\in(0,1)$, the following inequalities hold:

(i) Ratio-type reverse inequality

$$a^{1-\lambda}b^{\lambda}\le(1-\lambda)a+\lambda b\le a^{1-\lambda}b^{\lambda}\exp\left\{\frac{\lambda(1-\lambda)(a-b)^2}{d_1^2}\right\}$$

where $d_1=\min\{a,b\}$

(ii) Difference-type reverse inequality

$$a^{1-\lambda}b^{\lambda}\le(1-\lambda)a+\lambda b\le a^{1-\lambda}b^{\lambda}+\lambda(1-\lambda)\log^2\left(\frac{a}{b}\right)d_2$$

where $d_2=\max\{a,b\}$

We recall a reverse scalar Young's inequality stated in [5] and [6] by F. Kittaneh and Y. Manasrah and then we will use it in the next section.

Proposition 1. If $a,b>0$ and $\lambda\in(0,1)$, then

$$\lambda a+(1-\lambda)b\le R_0\left(\sqrt{a}-\sqrt{b}\right)^2+a^{\lambda}b^{1-\lambda}$$

and

$$\left(\lambda a+(1-\lambda)b\right)^2\le R_0\left(a-b\right)^2+\left(a^{\lambda}b^{1-\lambda}\right)^2$$

where $R_0=\max\{\lambda,1-\lambda\}$.

The definition of the isotonic linear functionals appears in [1] and [2] and will be used also in the next section.

Definition 1. [1] Let E be a nonempty set and L be a class of real-valued functions $f:E\to\mathbb{R}$ having the following properties:

(L1) If $f,g\in L$ and $a,b\in\mathbb{R}$ the $(af+bg)\in L$

(L2) $1\in L$ i.e. if $f(t)=1$ for all $t\in E$, then $f\in L$

An isotonic linear functional is a functional $A:L\to\mathbb{R}$ having the following properties:

(A1) If $f,g\in L$ and $a,b\in\mathbb{R}$ then $A(af+bg)=aA(f)+bA(g)$.

(A2) If $f\in L$ and $f(t)\ge 0$ for all $t\in E$ then $A(f)\ge 0$.

The mapping A is said to be normalized if

(A3) $A(1)=1$.

In order to give some examples for the main results of this paper it is necessary to enunciate Theorem 3.2 from [1].

Theorem 2. [1] Let $\mathbb{T}$ be a time scale. For $a,b\in\mathbb{T}$ with $a<b$, let

$$E=[a,b)\cap\mathbb{T},\ L=C_{rd}\left([a,b),\mathbb{R}\right)$$

Then (L1) and (L2) are satisfied. Moreover, let

$$A(f)=\int_a^b f(t)\Delta t$$

where the integral is the Cauchy delta time - scale integral. Then (A1) and (A2) are satisfied.

2. MAIN RESULTS

The following result is a generalization of Holder's inequality for isotonic linear functionals under some additional conditions.

Theorem 3. Let $\lambda\in(0,1)$, E a nonempty set, L satisfying conditions (L1), (L2) and A satisfying conditions (A1) and (A2) on the set E. If f and g are two positive functions with $f,g,f^{\frac{1}{2}},g^{\frac{1}{2}},\ f^{\lambda}g^{1-\lambda},\left(A(g)\log f-A(f)\log g\right)^2\in L$ and $f(x)\geq A(f)$ and $g(x)\geq A(g)$, $(\forall)x\in E$ then

$$2r-1+A_1(\lambda).A\left[\left(A(g)\log f-A(f)\log g\right)^2\right]\leq 2r\frac{A\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)}-\frac{A\left(f^{\lambda}g^{1-\lambda}\right)}{A^{\lambda}(f)A^{1-\lambda}(g)}$$

and

$$2(1-r)\frac{A\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)}-\frac{A\left(f^{\lambda}g^{1-\lambda}\right)}{A^{\lambda}(f)A^{1-\lambda}(g)}\leq 1-2r+B_1(\lambda).A\left[\left(A(g)\log f-A(f)\log g\right)^2\right]$$

where $r=\min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$.

Proof: Taking into account the inequalities from Theorem 1, where we use $A_1(\lambda)$ instead of $A(\lambda)$ and $B_1(\lambda)$ instead of $B(\lambda)$, for $a=\frac{f}{A(f)}$ and $b=\frac{g}{A(g)}$ we will have:

$$r\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2+A_1(\lambda).\log^2\left(\frac{f}{A(f)}\frac{A(g)}{g}\right)\leq\frac{f}{\lambda^{A(f)}}+(1-\lambda)\frac{g}{A(g)}-\frac{f^{\lambda}g^{1-\lambda}}{A^{\lambda}(f)A^{1-\lambda}(g)}\leq$$

$$\leq(1-r)\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2+B_1(\lambda).\log^2\left(\frac{f}{A(f)}\frac{A(g)}{g}\right)$$

By calculus these inequalities can be written as:

$$r\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^{2}+A_1(\lambda).\left(A(g)\log f-A(f)\log g\right)^{2}\le$$

$$\le\lambda\frac{f}{A(f)}+(1-\lambda)\frac{g}{A(g)}-\frac{f^{\lambda}g^{1-\lambda}}{A^{\lambda}(f)A^{1-\lambda}(g)}\le$$

$$\le(1-r)\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^{2}+B_1(\lambda).\left(A(g)\log f-A(f)\log g\right)^{2}$$

Applying now condition (A2) from Definition 1 we obtain,

$$rA\left[\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^{2}\right]+A_1(\lambda).A\left[\left(A(g)\log f-A(f)\log g\right)^{2}\right]\le$$

$$\le1-\frac{A\left(f^{\lambda}g^{1-\lambda}\right)}{A^{\lambda}(f)A^{1-\lambda}(g)}\le(1-r)A\left[\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^{2}\right]+B_1(\lambda).A\left[\left(A(g)\log f-A(f)\log g\right)^{2}\right]$$

or

$$2r\left(1-\frac{A\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)}\right)+A_1(\lambda).A\left[\left(A(g)\log f-A(f)\log g\right)^{2}\right]\le$$

$$\le1-\frac{A\left(f^{\lambda}g^{1-\lambda}\right)}{A^{\lambda}(f)A^{1-\lambda}(g)}\le2(1-r)\left(1-\frac{A\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)}\right)+B_1(\lambda).A\left[\left(A(g)\log f-A(f)\log g\right)^{2}\right]$$

which implies inequalities of the theorem.

Considering now other additional conditions we can give also the following generalization of Holder's inequality for isotonic linear functionals.

Theorem 4. Let $\lambda\in(0,1)$, E a nonempty set, L satisfying conditions (L1), (L2) and A satisfying conditions (A1) and (A2) on the set E. If $f,g,f^{\frac{1}{2}},g^{\frac{1}{2}}$, $f^{\lambda}g^{1-\lambda}$, $fg.\log^{2}\left(\frac{fA(g)}{gA(f)}\right)\in L$ and $0<f(x)\le A(f)$ and $0<g(x)\le A(g)$, $(\forall)x\in E$ then

$$2r\left(1-\frac{A\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)}\right)+A_1(\lambda).\frac{A\left[fg.\log^2\left(\frac{fA(g)}{gA(f)}\right)\right]}{A(f)A(g)}\le 1-\frac{A\left(f^{\lambda}g^{1-\lambda}\right)}{A^{\lambda}(f)A^{1-\lambda}(g)}\le$$

$$\le 1(1-r)\left(1-\frac{A\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)}\right)+B_1(\lambda).\frac{A\left[fg.\log^2\left(\frac{fA(g)}{gA(f)}\right)\right]}{A(f)A(g)}$$

where $r=\min\{\lambda,1-\lambda\}$, $A_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{r}{4}$ and $B_1(\lambda)=\frac{\lambda(1-\lambda)}{2}-\frac{1-r}{4}$.

Proof: Using the hypothesis and inequalities from Application 3.2 for $a=\frac{f}{A(f)}$ and $b=\frac{g}{A(g)}$ we will obtain when $A(\lambda)$ from Application 3.2 is replaced by $A_1(\lambda)$ and $B(\lambda)$ by $B_1(\lambda)$ the following inequalities:

$$r\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2+A_1(\lambda)\frac{fg}{A(f)A(g)}.\log^2\left(\frac{f}{A(f)}\frac{A(g)}{g}\right)\le$$

$$\le\lambda\frac{f}{A(f)}+(1-\lambda)\frac{g}{A(g)}-\frac{f^{\lambda}g^{1-\lambda}}{A^{\lambda}(f)A^{1-\lambda}(g)}\le$$

$$\le(1-r)\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2+B_1(\lambda)\frac{fg}{A(f)A(g)}.\log^2\left(\frac{f}{A(f)}\frac{A(g)}{g}\right)$$

By condition (A2) from Definition 1 we get

$$rA\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)^2+A_1(\lambda)A\left[\frac{fg}{A(f)A(g)}.\log^2\left(\frac{f}{A(f)}\frac{A(g)}{g}\right)\right]\le$$

$$\le A\left[\lambda\frac{f}{A(f)}+(1-\lambda)\frac{g}{A(g)}\right]-\frac{A\left[f^{\lambda}g^{1-\lambda}\right]}{A^{\lambda}(f)A^{1-\lambda}(g)}\le$$

$$\le(1-r)A\left[\left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)}-\frac{g^{\frac{1}{2}}}{A^{\frac{1}{2}}(g)}\right)\right]^2+B_1(\lambda)A\left[\frac{fg}{A(f)A(g)}.\log^2\left(\frac{f}{A(f)}\frac{A(g)}{g}\right)\right]$$

and from here the desired inequalities.

A reverse of Holder's inequality under additional conditions for isotonic linear functionals will be given also below.

Theorem 5. Let $\lambda \in (0,1)$, E a nonempty set, L satisfying conditions (L1), (L2) and A satisfying conditions (A1) and (A2) on the set E. If *f* and *g* are two positive functions with f,g, $f^{\lambda}g^{1-\lambda}$, $f.\log^2\left(\frac{fA(g)}{gA(f)}\right) \in L$ and $\frac{f(x)}{A(f)} \geq \frac{g(x)}{A(g)}$, $(\forall) x \in E$ then

$$1 \leq \frac{A\left(f^{1-\lambda}g^{\lambda}\right)}{A^{1-\lambda}(f)A^{\lambda}(g)} + \frac{\lambda(1-\lambda)}{A(f)}.A\left[f.\log^2\left(\frac{fA(g)}{gA(f)}\right)\right]$$

Proof: In this case we will use the hypothesis and inequalities from Corollary 1 (ii) for $a = \frac{f}{A(f)}$ and obtaining:

$$\frac{(1-\lambda)f}{A(f)} + \lambda\frac{g}{A(g)} \leq \left(\frac{f}{A(f)}\right)^{1-\lambda}\left(\frac{g}{A(g)}\right)^{\lambda} + \lambda(1-\lambda)\left[\log^2\left(\frac{\frac{f}{A(f)}}{\frac{g}{A(g)}}\right)\right]\frac{f(x)}{A(f)}$$

and from condition (A2) from Definition 1 we obtain,

$$1 \leq \frac{A\left(f^{1-\lambda}g^{\lambda}\right)}{A^{1-\lambda}(f)A^{\lambda}(g)} + \frac{\lambda(1-\lambda)}{A(f)}.A\left[f.\log^2\left(\frac{fA(g)}{gA(f)}\right)\right]$$

Remark 1. If we consider in Theorem 3, Theorem 4 and Theorem 5 instead of the positive functions *f* and *g* the functions |*f*| and |*g*| then the corresponding inequalities remain true.

Example 1. If $\mathbb{T} = \mathbb{R}$ in Theorem 3.2 from [1] then $L = C([a,b],\mathbb{R})$ and $A(f) = \int_a^b f(t)dt$ and therefore inequalities from Theorem 3, Theorem 4 and Theorem 5 can be rewritten for these functionals under conditions of these theorems.

Example 2. If $\mathbb{T} = \mathbb{Z}$ in Theorem 3.2 from [1] then L consists of all real-valued functions defined on $[a,b-1] \cap \mathbb{Z}$ and $A(f) = \sum_{t=a}^{b-1} f(t)$ and therefore inequalities from Theorem 3, Theorem 4 and Theorem 5 can be rewritten for these functionals under conditions of these theorems.

In this case, Theorem 3 becomes:

If *f* and *g* are two positive functions, $f \geq A(f)$ and $g \geq A(g)$ then

$$2r-1+A_1(\lambda).\sum_{t=a}^{b-1}\left[\log f\sum_{t=a}^{b-1}g(t)-\log g\sum_{t=a}^{b-1}f(t)\right]^2(t)\le$$

$$\le 2r\frac{\sum_{t=a}^{b-1}\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)(t)}{\left(\sum_{t=a}^{b-1}f(t)\right)^{\frac{1}{2}}\left(\sum_{t=a}^{b-1}g(t)\right)^{\frac{1}{2}}}-\frac{\sum_{t=a}^{b-1}\left(f^{\lambda}g^{1-\lambda}\right)(t)}{\left(\sum_{t=a}^{b-1}f(t)\right)^{\lambda}\left(\sum_{t=a}^{b-1}g(t)\right)^{1-\lambda}}$$

and

$$2(1-r)\frac{\sum_{t=a}^{b-1}\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)(t)}{\left(\sum_{t=a}^{b-1}f(t)\right)^{\frac{1}{2}}\left(\sum_{t=a}^{b-1}g(t)\right)^{\frac{1}{2}}}-\frac{\sum_{t=a}^{b-1}\left(f^{\lambda}g^{1-\lambda}\right)(t)}{\left(\sum_{t=a}^{b-1}f(t)\right)^{\lambda}\left(\sum_{t=a}^{b-1}g(t)\right)^{1-\lambda}}\le$$

$$\le 1-2r+B_1(\lambda).\sum_{t=a}^{b-1}\left[\log f\sum_{t=a}^{b-1}g(t)-\log g\sum_{t=a}^{b-1}f(t)\right]^2(t)$$

Example 3. Let $h>0$. If $\mathbb{T}=h\,\mathbb{Z}$ in Theorem 3.2 from [1] then L consists of all real-valued functions defined on $[a,b-h]\cap h\,\mathbb{Z}$ and $A(f)=h\sum_{k=\frac{a}{h}}^{\frac{b}{h}-1}f(k)$ and therefore inequalities from Theorem 3, Theorem 4 and Theorem 5 can be rewritten for these functionals under conditions of these theorems.

Remark 2. Let $\lambda\in(0,1)$, E a nonempty set, L satisfying conditions (L1), (L2) and A and B satisfying conditions (A1) and (A2) on the set E. If *f* and *g* are two positive functions (or we can take |*f*| instead of *f* and |*g*| instead of *g*) with f,g, $f^{\frac{1}{2}},g^{\frac{1}{2}}$, $f^{\lambda}g^{1-\lambda}\in L$ then

$$\lambda+(1-\lambda)\frac{A(g)}{B(g)}+2R_0\frac{A\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)}{A^{\frac{1}{2}}(f)A^{\frac{1}{2}}(g)}\le R_0\left(1+\frac{A(g)}{B(g)}\right)+\frac{A\left(f^{\lambda}g^{1-\lambda}\right)}{A^{\lambda}(f)A^{1-\lambda}(g)}$$

and

$$1+(1-\lambda)\frac{A(g)}{B(g)}+\lambda\frac{B(f)}{A(f)}+2R_0\frac{A\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)+B\left(f^{\frac{1}{2}}g^{\frac{1}{2}}\right)}{A^{\frac{1}{2}}(f)B^{\frac{1}{2}}(g)}\le R_0\left(2+\frac{A(g)}{B(g)}+\frac{B(f)}{A(f)}\right)+\frac{A\left(f^{\lambda}g^{1-\lambda}\right)+B\left(f^{\lambda}g^{1-\lambda}\right)}{A^{\lambda}(f)B^{1-\lambda}(g)}$$

where $R_0=\max\{\lambda,1-\lambda\}$.

Proof: Taking into account the first inequality from Proposition 1, used for $a = \frac{f}{A(f)}$ and $b = \frac{g}{A(g)}$ we will have:

$$\lambda \frac{f}{A(f)} + (1-\lambda)\frac{g}{A(g)} \le R_0 \left(\frac{f^{\frac{1}{2}}}{A^{\frac{1}{2}}(f)} - \frac{g^{\frac{1}{2}}}{B^{\frac{1}{2}}(g)} \right)^2 + \frac{f^{\lambda} g^{1-\lambda}}{A^{\lambda}(f) B^{1-\lambda}(g)}$$

By Definition 1, the condition (A2), applied successively for A and then for B, we have:

$$\lambda + (1-\lambda)\frac{A(g)}{B(g)} \le R_0 \left(1 + \frac{A(g)}{B(g)} - 2\frac{A\left(f^{\frac{1}{2}} g^{\frac{1}{2}} \right)}{A^{\frac{1}{2}}(f) B^{\frac{1}{2}}(g)} \right) + \frac{A\left(f^{\lambda} g^{1-\lambda} \right)}{A^{\lambda}(f) B^{1-\lambda}(g)}$$

and

$$\lambda \frac{B(f)}{A(f)} + (1-\lambda) \le R_0 \left(\frac{B(f)}{A(f)} + 1 - 2\frac{B\left(f^{\frac{1}{2}} g^{\frac{1}{2}} \right)}{A^{\frac{1}{2}}(f) B^{\frac{1}{2}}(g)} \right) + \frac{B\left(f^{\lambda} g^{1-\lambda} \right)}{A^{\lambda}(f) B^{1-\lambda}(g)}$$

and summing now these two inequalities we find the desired inequality.

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