ORIGINAL PAPER

AN ALGORITHM OF HIGHER ORDER ACCURACY FOR THE NUMERICAL SOLUTION OF GENERAL TWO-POINT BOUNDARY VALUE PROBLEMS

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Abstract. *The aim of this paper is to describe new efficient method for solving general two-point boundary value problems subject to first boundary conditions. We present the construction of non-standard algorithm. The order of the method is six. This new algorithm is suitable for solving different kind of problems. Numerical examples are presented to illustrate the efficiency of the method, to demonstrate the accuracy of the method. Results are compared with the exact solution.*

Keywords: *Non-classical finite difference method, Boundary value problems, Six order method, Second order general differential equation, Rational approximation.*

2010 AMS Classification: *65L05, 65L12.*

1. INTRODUCTION

A nonlinear ordinary differential equations are used to model different kind of problems in all branches of engineering and sciences. The ideal scenario is to obtain analytical solutions but in most of the cases, this is not possible to achieve. Therefore, it is necessary to turn to numerical methods to obtain numerical solution.

In this work, new approach is considered for the development of the numerical method for the numerical solution of problems of the form

$$y''(x) = f(x, y, y'), \qquad y(a) = \alpha, \qquad y(b) = \beta \tag{1}$$

where α, β are real finite constants , $[a,b] \subset \mathbb{R}$, $y(x)$, $f(x, y, y') \in \mathbb{R}$.

A literature regarding this numerical solution of the two-point boundary value problem is given in [1, 2]. Much research has been done on the numerical solution of nonlinear boundary value problems. Methods for the numerical solution can be separated into two groups, the iterative and non-iterative methods. Usually, non-iterative methods adopted for linear whereas iterative methods for non-linear boundary value problems. An iterative numerical solution, a finite difference method of order six is given in [3].

The existence and uniqueness of the solution for the problem (1) is assumed. The specific assumption on *f (x, y, y')*, to ensure existence and uniqueness will not be considered [1, 4].

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We develop a different kind of finite difference formula and discretization method based on power series expansion (Taylor's and binomial) as an alternative to the standard method in [3]. A numerical method is called non classical rational approximation method i.e. non- standard finite difference method [5, 6]. A numerical comparison between the proposed method and Chawla's [3], to demonstrate the effectiveness of the method is given.

This work is organized as follows. In section 2, we describe the development of the proposed numerical method and outlined the derivation of the method in section 3. Computational results are reported in order to show its computational advantages in section 4. Conclusion and discussion on the performance of the proposed method in final section 5.

2. DEVELOPMENT AND DERIVATION OF THE METHOD

The classical methods generally perform poorly when it deals with unconventional problem [5]. It is equally true when specifically designed schemes for unconventional problems do not perform well on conventional problems [7]. A nonlinear multistep rational finite difference method is designed for conventional problems, the development of which is unconventional. Let $h=(b-a)/N$ be uniform step length and *N* be positive integer. Thus we have *N+1* nodes such that $x_i = a + i.h, i = 0(1)N$. Let the exact solution $y(x)$ at $x = x_i$ is denoted by y_i. Let us denote f_i as the approximation of the theoretical value of the force function $f(x, y, y')$ at node $x = x_i$, $i = 1(1)N$. We can define other notations used in this article i.e. $f_{i\pm1}$ and $y_{i\pm1}$ in the similar way. Also we define $x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$ and $y''_{i+\frac{1}{2}} = f\left(x_{i+\frac{1}{2}}, y_{i+\frac{1}{2}}, y'_{i+\frac{1}{2}}\right) = f_{i+\frac{1}{2}}$ … etc for $i = 0(1)N-1$. Suppose we have to determine a number y_i, which is an approximation to the numerical value of the theoretical solution $y(x)$ of the problem (1) at the nodal point x_i, $i = 1(1)N-1$. Assuming local assumption that no previous truncation errors have been made [8] i.e. $y_{i-1} = y(x_{i-1})$. Let us define following approximations [3],

$$\bar{y}'_i = \frac{(y_{i+1} - y_{i-1})}{2h} \tag{2}$$

$$\bar{y}'_{i+1} = \frac{(3y_{i+1} - 4y_i + y_{i-1})}{2h} \tag{3}$$

$$\bar{y}'_{i-1} = \frac{(3y_{i-1} - 4y_i + y_{i+1})}{2h} \tag{4}$$

$$\overline{f_i} = f\left(x_i, y_i, \bar{y}'_i\right) \tag{5}$$

$$\overline{f_{i+1}} = f\left(x_{i+1}, y_{i+1}, \bar{y}'_{i+1}\right) \tag{6}$$

$$\overline{f_{i-1}} = f\left(x_{i-1}, y_{i-1}, \overline{y}'_{i-1}\right) \tag{7}$$

$$\overline{y}_{i+\frac{1}{2}} = \frac{\left(15y_{i+1} + 18y_i - y_{i-1}\right)}{32} - \frac{h^2\left(3\overline{f}_{i+1} + 4\overline{f}_i - \overline{f}_{i-1}\right)}{64} \tag{8}$$

$$\overline{y}_{i-\frac{1}{2}} = \frac{\left(15y_{i-1} + 18y_i - y_{i+1}\right)}{32} - \frac{h^2\left(3\overline{f}_{i-1} + 4\overline{f}_i - \overline{f}_{i+1}\right)}{64} \tag{9}$$

$$\overline{\overline{y}}'_{i+1} = \frac{\left(y_{i+1} - y_{i-1}\right)}{2h} + \frac{h\left(2\overline{f}_i + \overline{f}_{i+1}\right)}{3} \tag{10}$$

$$\overline{\overline{y}}'_{i-1} = \frac{\left(y_{i+1} - y_{i-1}\right)}{2h} - \frac{h\left(2\overline{f}_i + \overline{f}_{i+1}\right)}{3} \tag{11}$$

$$\overline{\overline{y}}'_{i+\frac{1}{2}} = \frac{\left(5y_{i+1} - 6y_i + y_{i-1}\right)}{4h} - \frac{h\left(3\overline{f}_{i+1} + 8\overline{f}_i + \overline{f}_{i-1}\right)}{48} \tag{12}$$

$$\overline{\overline{y}}'_{i-\frac{1}{2}} = \frac{\left(-5y_{i+1} + 6y_i - y_{i-1}\right)}{4h} + \frac{h\left(3\overline{f}_{i-1} + 8\overline{f}_i + \overline{f}_{i+1}\right)}{48} \tag{13}$$

$$\overline{\overline{f}}_{i+1} = f\left(x_{i+1}, y_{i+1}, \overline{\overline{y}}'_{i+1}\right) \tag{14}$$

$$\overline{\overline{f}}_{i-1} = f\left(x_{i-1}, y_{i-1}, \overline{\overline{y}}'_{i-1}\right) \tag{15}$$

$$\overline{\overline{f}}_{i+\frac{1}{2}} = f\left(x_{i+\frac{1}{2}}, \overline{y}_{i+\frac{1}{2}}, \overline{\overline{y}}'_{i+\frac{1}{2}}\right) \tag{16}$$

$$\overline{\overline{f}}_{i-\frac{1}{2}} = f\left(x_{i-\frac{1}{2}}, \overline{y}_{i-\frac{1}{2}}, \overline{\overline{y}}'_{i-\frac{1}{2}}\right) \tag{17}$$

Finally let us define following approximations,

$$\hat{y}'_i = \overline{y}'_i + h\left(a_1\left(\overline{f}_{i+1} - \overline{f}_{i-1}\right) + a_2\left(\overline{\overline{f}}_{i+\frac{1}{2}} - \overline{\overline{f}}_{i-\frac{1}{2}}\right)\right) \tag{19}$$

$$\overline{\overline{y}}'_i = \overline{y}'_i + h\left(a_6\left(\overline{\overline{f}}_{i+1} - \overline{\overline{f}}_{i-1}\right) + a_7\left(\overline{\overline{f}}_{i+\frac{1}{2}} - \overline{\overline{f}}_{i-\frac{1}{2}}\right)\right) \tag{20}$$

$$\hat{\hat{y}}'_i = \overline{y}'_i + h\left(a_3\left(\overline{f}_{i+1} - \overline{f}_{i-1}\right) + a_4\left(\overline{\overline{f}}_{i+1} - \overline{\overline{f}}_{i-1}\right) + a_5\left(\overline{\overline{f}}_{i+\frac{1}{2}} - \overline{\overline{f}}_{i-\frac{1}{2}}\right)\right) \tag{21}$$

where a_i, $i = 1(1)7$, are free parameters to be determined under appropriate conditions.

Set

$$\hat{f}_i = f\left(x_i, y_i, \hat{y}'_i\right) \tag{22}$$

$$\hat{\hat{f}}_i = f\left(x_i, y_i, \hat{\hat{y}}'_i\right) \tag{23}$$

$$\overline{\overline{f}}_i = f\left(x_i, y_i, \overline{\overline{y}}'_i\right) \tag{24}$$

Then at each internal mesh point x_i to discretize problems (1), bearing in mind the development of non- standard finite difference method [8] and following the ideas in [9, 10], we propose the rational finite difference scheme [11] as,

$$y_{i+1} - 2y_i + y_{i-1} = \frac{8h^2\left(\overline{\overline{f}}_{i+1} + 28\hat{f}_i + \overline{\overline{f}}_{i-1}\right)\left(\overline{\overline{f}}_i\right)^2}{240\left(\overline{\overline{f}}_i\right)^2 - 12\overline{\overline{f}}_i\left(\overline{\overline{f}}_{i+1} - 2\hat{f}_i + \overline{\overline{f}}_{i-1}\right) + \left(\overline{\overline{f}}_{i+1} - 2\hat{f}_i + \overline{\overline{f}}_{i-1}\right)^2} \tag{25}$$

Thus, the method consists in finding an approximation y_i for the theoretical solution $y(x_i)$, $i = 1(1)N-1$ of the problem (1) by solving the system $(N-1)\times(N-1)$ nonlinear equations (25). The system of nonlinear equations (13) can be solved using Newton – Raphson method.

3. THE DERIVATION OF THE METHOD

In this section we outline the derivation of the method (25). From (4), expand $\overline{y}'_{i-1}$ in a Taylor series about the node x_{i-1} and simplify the expansion, we have

$$\overline{y}'_{i-1} = y'_{i-1} + O\left(h^2\right) \tag{26}$$

Thus $\overline{y}'_{i-1}$ provide an $O\left(h^2\right)$ approximation for y'_{i-1}. Similarly from (2) and (3), we have

$$\overline{y}'_i = y'_i + O\left(h^2\right) \tag{27}$$

$$\overline{y}'_i = y'_i + O\left(h^2\right) \tag{28}$$

So from (7) and (26),

$$\overline{f}_{i-1} = f\left(x_{i-1}, y_{i-1}, y'_{i-1} + O\left(h^2\right)\right) = f\left(x_{i-1}, y_{i-1}, y'_{i-1}\right) + O\left(h^2\right) \tag{29}$$

Thus $\overline{f}_{i-1}$ provides an $O\left(h^2\right)$ approximation for f_{i-1}. Similarly we can define other approximations of the forcing functions of different order. Using the approximation of $\overline{\overline{f}}_{i+1}$, $\overline{\overline{f}}_{i}$ and $\overline{\overline{f}}_{i-1}$, we can prove that $\overline{\overline{f}}_{i+1} + 28\overline{\overline{f}}_{i} + \overline{\overline{f}}_{i-1}$ will provide an $O\left(h^6\right)$ approximation for $f_{i+1} + 28 f_i + f_{i-1}$, if

$$7a_i = a_2 = -\frac{1}{12}$$

i.e.

$$\overline{\overline{f}}_{i+1} + 28\overline{\overline{f}}_{i} + \overline{\overline{f}}_{i-1} = f_{i+1} + 28 f_i + f_{i-1} + O\left(h^6\right) \tag{30}$$

Similarly we can prove that $\overline{\overline{f}}_{i+1} - 2\overline{\overline{f}}_{i} + \overline{\overline{f}}_{i-1}$ will provide an $O\left(h^6\right)$ approximation for $f_{i+1} - 2 f_i + f_{i-1}$, if

$$a_3 = \frac{1}{6}, \qquad a_4 = -\frac{1}{12}, \qquad a_5 = -\frac{4}{12}$$

i.e.

$$\overline{\overline{f}}_{i+1} - 2\overline{\overline{f}}_{i} + \overline{\overline{f}}_{i-1} = f_{i+1} - 2 f_i + f_{i-1} + O\left(h^6\right) \tag{31}$$

and $\overline{\overline{f}}_{i}$ will provide an $O\left(h^6\right)$ approximation for f_i if

$$a_6 = -\frac{1}{180} \quad \text{and} \quad a_7 = \frac{7}{45}$$

i.e.

$$\overline{\overline{f}}_{i} = f_i + O\left(h^6\right) \tag{32}$$

Thus by the application of approximations (30, 31, 32) and method [11], we have our proposed rational approximation method for numerical solution of problem (1) at each interior nodal point x_i, $i = 1(1)N-1$ as,

$$y_{i+1} - 2y_i + y_{i-1} = \frac{8h^2\left(\overline{\overline{f}}_{i+1} + 28\overline{\overline{f}}_{i} + \overline{\overline{f}}_{i-1}\right)\left(\overline{\overline{f}}_{i}\right)^2}{240\left(\overline{\overline{f}}_{i}\right)^2 - 12\overline{\overline{f}}_{i}\left(\overline{\overline{f}}_{i+1} - 2\overline{\overline{f}}_{i} + \overline{\overline{f}}_{i-1}\right) + \left(\overline{\overline{f}}_{i+1} - 2\overline{\overline{f}}_{i} + \overline{\overline{f}}_{i-1}\right)^2} \tag{33}$$

Thus, we have a reduced order computational method (33) for problems (1). But if we use approximations (30, 31, 32) and second order difference approximation for $f_{i+1} - 2 f_i + f_{i-1}$ in (33), we will obtained following method,

$$y_{i+1} - 2y_i + y_{i-1} = \frac{8h^2\left(y''_{i+1} + 28y''_i + y''_{i-1}\right)\left(y''_i\right)^2}{240\left(y''_i\right)^2 - 12h^2 y''_i\, y_i^{(4)} + h^4\left(y_i^{(4)}\right)^2} \tag{34}$$

where $y''_i = f_i$ and $y_i^{(4)} = f''_i$ … etc. Thus the theoretical order of accuracy of method (34) is six.

4. NUMERICAL EXPERIMENTS

In this section, the proposed method (33) is applied to solve three different model problems. We have used Newton-Raphson iteration method to solve the system of nonlinear equations arises from equation (33). All computations were performed on a MS Window 2007 professional operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Duo Core 2.20 Ghz PC. Let y_i, the numerical value calculated by formulae (33), an approximate value of the theoretical solution $y(x)$ at the grid point $x = x_i$. The maximum absolute error

$$MAE(y) = \max_{1<i<N-1} \left| y(x_i) - y_i \right|$$

are shown in Tables 1-3, for different value of *h*, the step length. Also we have computed y_i, using formula given by Chawla [3] and shown in the tables.

Example 1. Consider the following two- point boundary value problem

$$y''(x) = \frac{\left(y(x) + xy'(x)\right)}{(1+x)}, \qquad 0 < x < 1$$

with the boundary conditions $y(1) = e, \quad y(0) = 1$. In Table-1, the maximum absolute error presented in exact solution $y(x) = e^x$.

Example 2. Consider the following nonlinear two-point boundary value problem

$$y''(x) = \frac{\left(e^{2y} + (y')^2\right)}{2.0}, \qquad 0 < x < 1$$

with the boundary conditions $y(0) = 0, \quad y(1) = -\log(2.0)$. In Table-2, the MAE presented in exact solution $y(x) = \log\left(\frac{1.0}{1+x}\right)$.

Example 3. Consider the following nonlinear two-point boundary value problem

$$y''(x) = -\frac{x}{\sqrt{1-y}}\, y' + f(x), \qquad 0 < x < 1$$

with the boundary conditions $y(0) = 0, \quad y(1) = -3$. In Table-3, the MAE presented in exact solution $y(x) = 1 - \left(x^2 + 1\right)^2$.

Table 1.Maximum absolute error $y(x)=e^x$ **in for example 1.**

Error		Step length (h)		
		4	8	16
(33)	MAE	.20265579(-5)	.23841858(-6)	No change
Chawla's	MAE	.23841858(-6)	No change	No change

Table 2. Maximum absolute error in $y(x)=\log\left(\frac{1.0}{1+x}\right)$ **for example 2.**

Error		Step length (h)			
		4	8	16	32
(33)	MAE	.20310283(-4)	.12814999(-5)	.59604645(-7)	No change
Chawla's	MAE	.30264258(-4)	.77486038(-5)	.15497208(-5)	.59604645(-7)

Table 3.Maximum absolute error in $y(x)=1-\left(x^2+1\right)^2$ **for example 3.**

Error		Step length (h)		
		4	8	16
(33)	MAE	.10848045(-3)	.61988831(-5)	.37252903(-8)
Chawla's	MAE	.47683716(-6)	.37252903(-8)	No change

CONCLUSIONS

A new kind of finite difference scheme is presented for numerical solution of two-point boundary value problems. It follows from derivation and discussion in [11, 3] ,the method (34) is at least of sixth order. However it is not evident in computational results. The numerical results only confirm that method has order more than four. We obtain a comparable results using three functions evaluation at interior grid points unlike [3]. We cannot claim, in general that our method is better than standard finite difference method. But numerical results show that our method generate results of approximately same accuracy as that method in [3]. It isan alternative method, to get reliable results with less effort and obtain competitive results to those obtained with other methods.

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