ORIGINAL PAPER

ON MIXED TRILATERAL GENERATING RELATIONS FOR KONHAUSER BIORTHOGONAL POLYNOMIALS

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Abstract. *In this note, we have obtained some novel results on mixed trilateral generating relations involving the polynomials,* $Y_n^{\alpha-nk}(x;k)$*, a modified form of Konhauser biorthogonal polynomials,* $Y_n^{\alpha}(x;k)$ *by group theoretic method. As special cases, we have obtained the corresponding results on generalised Laguerre polynomials. Some applications of our results are also discussed.*

Keywords: *Laguerre polynomials, biorthogonal polynomials, mixed trilateral generating functions.*

AMS-2010 Subject Classification Code: *33C 45, 33C 47.*

1. INTRODUCTION

The polynomial sets $\{Y_n^{\alpha}(x;k)\}$ and $\{Z_n^{\alpha}(x;k)\}$, discussed by J.D.E. Konhauser [1-2], are biorthogonal with respect to the weight function $x^{\alpha}e^{-x}$ over the interval $(0,\infty)$, $\alpha > -1$, k is a positive integer. For k = 1, these polynomials reduce to the generalized Laguerre polynomials, $L_n^{\alpha}(x)$. An explicit expression for the polynomials $Y_n^{\alpha}(x;k)$ was given by Carlitz [3] in the following form:

$$Y_n^{\alpha}(x;k) = \frac{1}{n!}\sum_{i=0}^{n}\frac{x^i}{i!}\sum_{j=0}^{i}(-1)^j\binom{i}{j}\left(\frac{j+\alpha+1}{k}\right)_n$$

where $(a)_n$ is the pochhammer symbol defined by

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$$(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)} = \begin{cases} 1, & \text{if } n=0, a \neq 0 \\ a(a+1)\dots(a+n-1,) & \forall n \in \{1,2,3\dots\}. \end{cases}$$

In a recent paper [7], the present authors have proved the following theorem on bilateral generating relations involving the polynomials, $Y_n^{\alpha-nk}(x;k)$ a modified form of Konhauser biorthogonal polynomials, $Y_n^{\alpha}(x;k)$.

Theorem 1. If there exists a unilateral generating relation of the form

$$G(x,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha-nk}(x;k) w^n \tag{1.1}$$

then

$$(1+kt)^{\frac{(1+\alpha-k)}{k}} \exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, \frac{vt}{1+kt}\right) = \sum_{n=0}^{\infty} \sigma_n(v) Y_n^{\alpha-nk}(x;k) t^n \tag{1.2}$$

where

$$\sigma_n(v) = \sum_{p=0}^{n} a_p k^{n-p} \binom{n}{p} v^p .$$

The object of the present paper is to generalise the above bilateral generating relation into mixed trilateral generating relation by the group-theoretic method . A particular cases of interest is also discussed in this paper. The main results of our investigation are stated in the form of the following theorems:

Theorem 2. If there exists a bilateral generating relation of the form

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha-nk}(x;k) g_n(u) w^n , \tag{1.3}$$

where $g_n(u)$ is an arbitrary polynomial of degree n, then

$$(1+kt)^{\frac{(1+\alpha-k)}{k}} \exp\exp\left\{x - x(1+kt)^{\frac{1}{k}}\right\} G\left(x(1+kt)^{\frac{1}{k}}, u, \frac{vt}{1+kt}\right) = \sum_{n=0}^{\infty} \sigma(u,v) Y_n^{\alpha-nk}(x;k) t^n \tag{1.4}$$

where

$$\sigma_n(u,v) = \sum_{p=0}^{n} a_p k^{n-p} \binom{n}{p} g_p(u) v^p$$

Furthermore, we would like to point it out that we have given some applications of our theorem in this paper.

2. PROOF OF THEOREM 2

At first, we consider the following linear partial differential operator [7]:

$$R = xy\frac{\partial}{\partial x} - ky^2\frac{\partial}{\partial y} - (x + k - \alpha - 1)y$$

such that

$$R\left(Y_n^{\alpha-nk}(x;k)\,y^n\right) = k(n+1)Y_{n+1}^{\alpha-nk-k}(x;k)\,y^{n+1}. \tag{2.1}$$

The extended form of the group generated by R is given by

$$e^{wR}f(x,y) = (1+kwy)^{\frac{1+\alpha-k}{k}}\exp\left\{x - x(1+kwy)^{\frac{1}{k}}\right\}\times f\left(x(1+kwy)^{\frac{1}{k}}, \frac{y}{1+kwy}\right), \tag{2.2}$$

where $f(x,y)$ is an arbitrary function and w is an arbitrary constant.

Let us consider the generating relation of the form:

$$G(x,u,w) = \sum_{n=0}^{\infty} a_n Y_n^{\alpha-nk}(x;k)\,g(u)\,w^n. \tag{2.3}$$

Replacing w by wvy in the both sides of (2.3) we have

$$G(x,u,wvy) = \sum_{n=0}^{\infty} a_n\left(Y_n^{\alpha-nk}(x;k)\,g_n(u)\,y^n\right)(wv)^n. \tag{2.4}$$

Operating e^{wR} on both sides of (2.4), we get

$$e^{wR}\left(G(x,u,wvy)\right) = e^{wR}\left(\sum_{n=0}^{\infty} a_n\left(Y_n^{\alpha-nk}(x;k)\,g_n(u)\,y^n\right)(wv)^n\right). \tag{2.5}$$

Now the left member of (2.5), with the help of (2.2), reduces to

$$(1+kwy)^{\frac{1+\alpha-k}{k}}\exp\left\{x - x(1+kwy)^{\frac{1}{k}}\right\}G\left(x(1+kwy)^{\frac{1}{k}}, u, \frac{wvy}{1+kwy}\right). \tag{2.6}$$

The right member of (2.5), with the help of (2.1), becomes

$$= \sum_{n=0}^{\infty}\sum_{p=0}^{n} a_{n-p}\frac{w^p}{p!}k^p(n-p+1)_p\,Y_n^{\alpha-nk}(x;y)\,g_{n-p}(u)\,y^n(wv)^{n-p}. \tag{2.7}$$

Now equating (2.6) and (2.7) and then substituting $wy=t$ we get

$$(1+kt)^{\frac{1+\alpha-k}{k}}\exp\left\{x-x(1+kt)^{\frac{1}{k}}\right\}G\left(x(1+kt)^{\frac{1}{k}},u,\frac{vt}{1+kt}\right)=\sum_{n=0}^{\infty}Y_n^{\alpha-nk}(x;k)\sigma_n(u,v)t^n, \qquad (2.8)$$

where

$$\sigma_n(u,v)=\sum_{p=0}^{n}a_pk^{n-p}\binom{n}{p}g_n(u)v^p.$$

This completes the proof the theorem.

Special case. Now putting $k=1$ in our Theorem 2 we get the following result on generalised Laguerre polynomials:

Theorem 3. If there exists a bilateral generating relation of the form

$$G(x,u,w)=\sum_{n=0}^{\infty}a_nL_n^{(\alpha-n)}(x)g_n(u)w^n, \qquad (2.9)$$

where $g_n(u)$ is an arbitrary polynomial of degree $n,$ then

$$(1+t)^{\alpha}\exp(-xt)G\left(x(1+t),u,\frac{vt}{1+t}\right)=\sum_{n=0}^{\infty}\sigma_n(u,v)L_n^{(\alpha-n)}(x)t^n, \qquad (2.10)$$

where

$$\sigma_n(u,v)=\sum_{p=0}^{n}a_p\binom{n}{p}g_p(u)v^p,$$

which is also found derived in [5, 6].

3. APPLICATIONS

As an application of Theorem 2, we consider the following generating relation [4]:

$$\sum_{n=0}^{\infty}\frac{n!}{\Gamma(\beta+nl+1)}Y_n^{\alpha-nk}(x;k)Z_n^{\beta}(y;l)t^n=(1+t)^{\frac{1+\alpha-k}{k}}\exp\left\{x-x(1+t)^{\frac{1}{k}}\right\}H\left[x(1+t)^{\frac{1}{k}},\frac{-y^lt}{1+t}\right], \quad (3.1)$$

where

$$H[x,t]=\sum_{n=0}^{\infty}\frac{1}{\Gamma(\beta+nl+1)}Y_n^{\alpha-nk}(x;k)t^n.$$

If in our theorem, we take $a_n = \dfrac{n!}{\Gamma(\beta + nl + 1)}$, and $g_n(u) = Z_n^{\beta}(u;l)$ then

$$G(x,u,w) = (1+w)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1+w)^{\frac{1}{k}}\right\} H\left[x(1+w)^{\frac{1}{k}}, \frac{-u^l w}{1+w}\right].$$

Therefore by the application of our Theorem 2 we get the following generalization of the result (3.1):

$$\begin{aligned}&(1+kt+vt)^{\frac{1+\alpha-k}{k}} \exp\left\{x - x(1+kt+vt)^{\frac{1}{k}}\right\} X\ H\left[x(1+kt+vt)^{\frac{1}{k}}, \frac{-u^l vt}{1+kt+vt}\right] \\ &= \sum_{n=0}^{\infty} \sigma_n(u,v) Y_n^{\alpha-nk}(x;k) t^n, \end{aligned} \tag{3.2}$$

where

$$\sigma_n(u,v) = \sum_{p=0}^{n} a_p k^{n-p} \binom{n}{p} g_p(u) v^p.$$

It is of interest to mention that the result (3.2) for $k=1$ is also obtained by applying Theorem 3, on (3.1) for $k=1$.

4. CONCLUSIONS

From the above discussion, it is clear that whenever one knows a bilateral generating relation of the form (1.3, 2.9) then the corresponding mixed trilateral generating relation can at once be written down from (1.4, 2.10). So one can get a large number of mixed trilateral generating relations by attributing different suitable values to a_n in (1.3, 2.9).

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