ORIGINAL PAPER

POISSON SUM FORMULAE OF THE EXTENDED FRACTIONAL FOURIER TRANSFORM

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Abstract. *This paper studies the generalized Poisson summation formula from the extended fractional Fourier transform point of view and derived several new formulae associate with the extended fractional Fourier transform. The generalized Poisson sum formula is obtained based on the relationship of the extended fractional Fourier transform and the Fourier transform. Then we derived some results associated with the generalized Poisson sum formula.*

Keywords: *Fractional Fourier transform, extended fractional Fourier transform, Poisson sum formulae and band-limited signals.*

1. INTRODUCTION

The fractional Fourier transform is a generalization of the Fourier transform into fractional domains which are in between the time and frequency domains of the time-frequency plane. The fractional Fourier transform has received much attention in recent years. It has been applied in several areas, including optics and signal processing [1, 5, 6] and its relationship with the Fourier transform can be found in [7, 8]. The concept of Fourier transform of fractional ordered is introduced by Namias [4] in 1980 given by,

$$F^{\alpha}[f(t)](u) = \frac{e^{\left[i\left(\frac{\pi-2\alpha}{4}\right)\right]}}{\sqrt{2\pi sin\alpha}} \int_{-\infty}^{\infty} e^{-i\left[\frac{(t^2+u^2)}{2}cot\alpha - tucsc\alpha\right]} f(t)dt$$

The generalization of the fractional Fourier transform, which is known as extended fractional Fourier transform can be seen in [3] with two more parameters as,

$$F^{\alpha}_{a,b}[f(t)](u) = F^{\alpha}_{a,b}(u) = \int_{-\infty}^{\infty} e^{i\pi\left[(a^2t^2+b^2u^2)cot\alpha - 2abtucsc\alpha\right]} f(t)dt \tag{1.1}$$

In 2009 Bing-Zhao Li et. al. [2] had extended the Poisson sum formula and non-band-limited functions analysis associated with the fractional Fourier transform. In this paper we have generalized the traditional Poisson sum formula in the Fourier domain to the extended fractional Fourier transform domain and we proved some results for extended fractional Fourier transform.

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2. BAND-LIMITED FUNCTIONS IN EXTENDED FRACTIONAL FOURIER TRANSFORM DOMAIN

A function $f(t)$ is said to be band-limited with respect to Ω_α in extended fractional Fourier transform domain, when

$$F_{a,b}^{\alpha}(u) = 0 \quad \text{for} \quad |u| > \Omega_\alpha \tag{2.1}$$

where Ω_α is called the bandwidth of function $f(t)$ in the extended fractional Fourier transform domain and a function $f(t)$ is called the chirp-periodic with period τ and order α if it satisfies the following equation.

$$f(t)\, e^{i\pi a^2 t^2 \cot\alpha} = f(t+\tau) e^{i\pi a^2 (t+\tau)^2 \cot\alpha} \tag{2.2}$$

Without loss of generality, we assume $\sin\alpha > 0$ in the proceeding sections.

The following lemma gives the relation between extended fractional Fourier transform and Fourier transform.

Lemma 1: If $F_{a,b}^{\alpha}(u)$ is the extended fractional Fourier transform of a function $f(t)$ for order α, G is the Fourier transform operator and $g(t) = f(t)\, e^{i\pi a^2 t^2 \cot\alpha}$, then

$$F_{a,b}^{\alpha}(u) e^{-i\pi b^2 u^2 \cot\alpha} = G\left[g(t)\right](abu\csc\alpha) \tag{2.3}$$

$$F_{a,b}^{\alpha}\left(\frac{v\sin\alpha}{ab}\right) e^{-i\pi b^2 \left(\frac{v\sin\alpha}{ab}\right)^2 \cot\alpha} = G\left[g(t)\right](v) = G(v) \tag{2.4}$$

Proof: By (1.1)

$$F_{a,b}^{\alpha}(u) e^{-i\pi b^2 u^2 \cot\alpha} = \int_{-\infty}^{\infty} e^{-2i\pi(abu\csc\alpha)t} g(t)\, dt$$

where

$$g(t) = e^{i\pi a^2 t^2 \cot\alpha} f(t) \tag{2.5}$$

$$= \Im\left[g(t)\right](abu\csc\alpha)$$

Substituting $u \to \dfrac{v\sin\alpha}{ab}$

Therefore

$$F_{a,b}^{\alpha}\left(\frac{v\sin\alpha}{ab}\right) e^{-i\pi b^2 \left(\frac{v\sin\alpha}{ab}\right)^2 \cot\alpha} = \Im\left[g(t)\right](v) = G(v)$$

3. THE POISSON SUM FORMULAE IN THE EXTENDED FRACTIONAL FOURIER TRANSFORM DOMAIN

The Poisson sum formula demonstrates that the sum of infinite samples in time domain of a function $f(t)$ is equivalent to the sum of infinite samples of $F(u)$ in the extended fractional Fourier transform domain. The Poisson sum formula is as,

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} F\left(\frac{abn}{\tau}\right) e^{2i\pi a\left(\frac{abn}{\tau}\right)t} \tag{3.1}$$

or

$$\sum_{k=-\infty}^{\infty} f(k\tau) = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} F\left(\frac{abn}{\tau}\right) \tag{3.2}$$

where $F(u)$ is the traditional Fourier transform of function $f(t)$.

4. THE GENERALIZED POISSON SUM FORMULAE ASSOCIATED WITH THE EXTENDED FRACTIONAL FOURIER TRANSFORM

$F_{a,b}^{\alpha}(u)$ is the extended fractional Fourier of a function $f(t)$ for order α, the Poisson sum formula of function $f(t)$ associated with the extended fractional Fourier transform of order α can be derived as follows,

Theorem 1. The Poisson sum formula for a function $f(t)$ in the extended fractional Fourier domain of order α can be derived as in equations (4.1) and (4.2)

$$\begin{aligned}&\sum_{k=-\infty}^{\infty} f(t+k\tau) e^{i\pi a^2\left(2k\tau t+k^2\tau^2\right)\cot\alpha} \\ &= \frac{1}{\tau} e^{-i\pi a^2 t^2\cot\alpha}\sum_{n=-\infty}^{\infty} e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2\cot\alpha} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right) e^{2i\pi a\left(\frac{abn}{\tau}\right)t}\end{aligned} \tag{4.1}$$

and

$$\sum_{k=-\infty}^{\infty} f(k\tau) e^{i\pi a^2 k^2\tau^2\cot\alpha} = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2\cot\alpha} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right) \tag{4.2}$$

Proof: By (2.4)

$$F_{a,b}^{\alpha}\left(\frac{v\sin\alpha}{ab}\right) e^{-i\pi b^2\left(\frac{v\sin\alpha}{ab}\right)^2\cot\alpha} = \Im\left[g(t)\right](v) = G(v)$$

Replacing $v = \dfrac{abn}{\tau}$, making some manipulation we can get

$$\frac{1}{\tau}\sum_{n=-\infty}^{\infty} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right)e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2\cot\alpha+2i\pi a\left(\frac{abn}{\tau}\right)} = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} G\left(\frac{abn}{\tau}\right)e^{2i\pi a\left(\frac{abn}{\tau}\right)t} \tag{4.3}$$

Therefore by (3.1) and (2.5), (4.3) is as,

$$\frac{1}{\tau}\sum_{n=-\infty}^{\infty} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right)e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2\cot\alpha+2i\pi a\left(\frac{abn}{\tau}\right)} = \sum_{k=-\infty}^{\infty} f\left(t+k\tau\right)e^{i\pi a^2\left(t+k\tau\right)^2\cot\alpha}$$

Therefore

$$\sum_{k=-\infty}^{\infty} f\left(t+k\tau\right)e^{i\pi a^2\left(2k\tau t+k^2\tau^2\right)\cot\alpha}$$
$$=\frac{1}{\tau}e^{-i\pi a^2t^2\cot\alpha}\sum_{n=-\infty}^{\infty} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right)e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2\cot\alpha+2i\pi a\left(\frac{abn}{\tau}\right)t}$$

If we take $t=0$ in above equation then

$$\sum_{k=-\infty}^{\infty} f\left(k\tau\right)e^{i\pi a^2k^2\tau^2\cot\alpha} = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2\cot\alpha} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right)$$

Hence proved equations (4.1) and (4.2).

Theorem 2. If $f\left(t\right)$ is the band-limited function to Ω_α in the extended fractional Fourier transform domain of order α, then the Poisson sum formulae derived in theorem 1 can be rewritten as following according to the replica period τ.

i) When $\frac{\sin\alpha}{\tau} > \Omega_\alpha$, the Poisson sum reduces to the following:

$$\sum_{k=-\infty}^{\infty} f\left(t+k\tau\right)e^{i\pi a^2\left(2k\tau t+k^2\tau^2\right)\cot\alpha} = \frac{1}{\tau}e^{-i\pi a^2t^2\cot\alpha}F_{a,b}^{\alpha}\left(0\right) \tag{4.4}$$

ii) When $\frac{\Omega_\alpha}{2} < \frac{\sin\alpha}{\tau} < \Omega_\alpha$, the Poisson sum formula reduces to:

$$\sum_{k=-\infty}^{\infty} f\left(t+k\tau\right)e^{i\pi a^2\left(2k\tau t+k^2\tau^2\right)\cot\alpha}$$
$$=\frac{1}{\tau}e^{-i\pi a^2t^2\cot\alpha}\times\left\{F_{a,b}^{\alpha}\left(0\right)+e^{-i\pi b^2\left(\frac{\sin\alpha}{\tau}\right)^2\cot\alpha}\left[F_{a,b}^{\alpha}\left(\frac{-\sin\alpha}{\tau}\right)e^{-2i\pi a\left(\frac{ab}{\tau}\right)t}+F_{a,b}^{\alpha}\left(\frac{\sin\alpha}{\tau}\right)e^{2i\pi a\left(\frac{ab}{\tau}\right)t}\right]\right\} \tag{4.5}$$

iii) When $\frac{\Omega_\alpha}{n+1} < \frac{\sin\alpha}{\tau} < \frac{\Omega_\alpha}{n}$, the Poisson sum can be reduces to:

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) e^{i\pi a^2(2k\tau t + k^2\tau^2)\cot\alpha} = \frac{1}{\tau} e^{-i\pi a^2 t^2 \cot\alpha} \times \left\{ F_{a,b}^{\alpha}(0) + \sum_{k=1}^{n} e^{-i\pi b^2\left(\frac{k\sin\alpha}{\tau}\right)^2 \cot\alpha} \left[F_{a,b}^{\alpha}\left(\frac{-k\sin\alpha}{\tau}\right) e^{-2i\pi a\left(\frac{abk}{\tau}\right)t} + F_{a,b}^{\alpha}\left(\frac{k\sin\alpha}{\tau}\right) e^{2i\pi a\left(\frac{abk}{\tau}\right)t} \right] \right\} \tag{4.6}$$

Proof:

i) Since $f(t)$ is a Ω_α band-limited function in the extended fractional Fourier domain and $\frac{\sin\alpha}{\tau} > \Omega_\alpha$, and when $n \neq 0$ then right hand side of equation (4.1) will be

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) e^{i\pi a^2(2k\tau t + k^2\tau^2)\cot\alpha} = \frac{1}{\tau} e^{-i\pi a^2 t^2 \cot\alpha} \sum_{n=-\infty}^{\infty} e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2 \cot\alpha} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right) e^{2i\pi a\left(\frac{abn}{\tau}\right)t} = \frac{1}{\tau} e^{-i\pi a^2 t^2 \cot\alpha} F_{a,b}^{\alpha}(0)$$

ii) When $\frac{\Omega_\alpha}{2} < \frac{\sin\alpha}{\tau} < \Omega_\alpha$, then right hand side of equation (4.1) will be

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) e^{i\pi a^2(2k\tau t + k^2\tau^2)\cot\alpha} = \frac{1}{\tau} e^{-i\pi a^2 t^2 \cot\alpha} \sum_{n=-\infty}^{\infty} e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2 \cot\alpha} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right) e^{2i\pi a\left(\frac{abn}{\tau}\right)t}$$
$$= \frac{1}{\tau} e^{-i\pi a^2 t^2 \cot\alpha} \times \left\{ F_{a,b}^{\alpha}(0) + e^{-i\pi b^2\left(\frac{\sin\alpha}{\tau}\right)^2 \cot\alpha} \left[F_{a,b}^{\alpha}\left(\frac{-\sin\alpha}{\tau}\right) e^{-2i\pi a\left(\frac{ab}{\tau}\right)t} + F_{a,b}^{\alpha}\left(\frac{\sin\alpha}{\tau}\right) e^{2i\pi a\left(\frac{ab}{\tau}\right)t} \right] \right\}$$

iii) If we note $2n+1$ nonzero values of $F_{a,b}^{\alpha}(u)$, then summation part in right hand side of equation (4.1) will be,

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) e^{i\pi a^2(2k\tau t + k^2\tau^2)\cot\alpha} = \frac{1}{\tau} e^{-i\pi a^2 t^2 \cot\alpha} \sum_{n=-\infty}^{\infty} e^{-i\pi b^2\left(\frac{n\sin\alpha}{\tau}\right)^2 \cot\alpha} F_{a,b}^{\alpha}\left(\frac{n\sin\alpha}{\tau}\right) e^{2i\pi a\left(\frac{abn}{\tau}\right)t}$$
$$= \frac{1}{\tau} e^{-i\pi a^2 t^2 \cot\alpha} \times \left\{ F_{a,b}^{\alpha}(0) + \sum_{k=1}^{n} e^{-i\pi b^2\left(\frac{k\sin\alpha}{\tau}\right)^2 \cot\alpha} \left[F_{a,b}^{\alpha}\left(\frac{-k\sin\alpha}{\tau}\right) e^{-2i\pi a\left(\frac{abk}{\tau}\right)t} + F_{a,b}^{\alpha}\left(\frac{k\sin\alpha}{\tau}\right) e^{2i\pi a\left(\frac{abk}{\tau}\right)t} \right] \right\}$$

Theorem 3. If $F_{a,b}^{\alpha}(u)$ is the extended fractional Fourier of a band-limited function $f(t)$ with replica period τ and $f(t)$ multiplied with a chirp discretely as $e^{i\pi a^2 k^2 \tau^2 \cot\alpha} f(k\tau)$, then the sum of this formula is found to be finite as,

$$\sum_{k=-\infty}^{\infty} f(k\tau) e^{i\pi a^2 k^2 \tau^2 \cot\alpha}$$
$$= \begin{cases} \dfrac{1}{\tau} F_{a,b}^{\alpha}(0), & \dfrac{\sin\alpha}{\tau} > \Omega_\alpha \\ \dfrac{1}{\tau}\left\{ F_{a,b}^{\alpha}(0) + e^{-i\pi b^2 \left(\frac{\sin\alpha}{\tau}\right)^2 \cot\alpha} \left[F_{a,b}^{\alpha}\left(\dfrac{-\sin\alpha}{\tau}\right) + F_{a,b}^{\alpha}\left(\dfrac{\sin\alpha}{\tau}\right) \right] \right\}, & \dfrac{\Omega_\alpha}{2} < \dfrac{\sin\alpha}{\tau} < \Omega_\alpha \\ \dfrac{1}{\tau}\left\{ F_{a,b}^{\alpha}(0) + \sum_{k=1}^{n} e^{-i\pi b^2 \left(\frac{k\sin\alpha}{\tau}\right)^2 \cot\alpha} \left[F_{a,b}^{\alpha}\left(\dfrac{-k\sin\alpha}{\tau}\right) + F_{a,b}^{\alpha}\left(\dfrac{k\sin\alpha}{\tau}\right) \right] \right\}, & \dfrac{\Omega_\alpha}{n} < \dfrac{\sin\alpha}{\tau} < \dfrac{\Omega_\alpha}{n-1} \end{cases} \tag{4.7}$$

Proof: By taking $t = 0$ in equations (4.4), (4.5) and (4.6) we can simply get the equation (4.7).

Corollary 1. A band-limited function $f(t)$ in the extended fractional Fourier domain is sampled at period τ and multiplied with a chirp discretely as $e^{i\pi a^2 k^2 \tau^2 \cot\alpha} f(k\tau)$ if $\tau < \sin\alpha\ \Omega_\alpha$, then the discrete sum of $e^{i\pi a^2 k^2 \tau^2 \cot\alpha} f(k\tau)$ becomes identical to the integral of $e^{i\pi a^2 k^2 \tau^2 \cot\alpha} f(k\tau)$ as,

$$\sum_{k=-\infty}^{\infty} f(k\tau) e^{i\pi a^2 k^2 \tau^2 \cot\alpha} = \frac{1}{\tau} F_{a,b}^{\alpha}(0) = \frac{1}{\tau} \int_{-\infty}^{\infty} e^{i\pi a^2 k^2 \tau^2 \cot\alpha} f(t)\, dt \tag{4.8}$$

Proof: By taking parameter $u = 0$ in (1.1), then

$$F_{a,b}^{\alpha}(0) = \int_{-\infty}^{\infty} e^{i\pi a^2 t^2 \cot\alpha} f(t)\, dt \tag{4.9}$$

By (4.7) and (4.9)

$$\sum_{k=-\infty}^{\infty} f(k\tau) e^{i\pi a^2 k^2 \tau^2 \cot\alpha} = \int_{-\infty}^{\infty} e^{i\pi a^2 t^2 \cot\alpha} f(t)\, dt$$

Hence proved.

Corollary 2. When $\alpha = \dfrac{\pi}{2}$ the results derived in theorems 3 and 4 reduces to the well known results in the Fourier domain, in which $a = b = 1$.

Proof: If we replace the equations (4.1), (4.2), (4.4), (4.5) and (4.6) by $\alpha = \frac{\pi}{2}$ and $a = b = 1$, we can get (4.10), (4.11), (4.12), (4.13) and (4.14) respectively as,

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} F_{a,b}^{\frac{\pi}{2}}\left(\frac{n}{\tau}\right)e^{2i\pi\left(\frac{n}{\tau}\right)t} \tag{4.10}$$

$$\sum_{k=-\infty}^{\infty} f(k\tau) = \frac{1}{\tau}\sum_{n=-\infty}^{\infty} F_{a,b}^{\frac{\pi}{2}}\left(\frac{n}{\tau}\right) \tag{4.11}$$

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) = \frac{1}{\tau}F_{a,b}^{\frac{\pi}{2}}(0) \tag{4.12}$$

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) = \frac{1}{\tau}\sum_{n=-\infty}^{\infty}\left\{F_{a,b}^{\frac{\pi}{2}}(0) + F_{a,b}^{\frac{\pi}{2}}\left(\frac{-1}{\tau}\right)e^{-2i\pi\left(\frac{1}{\tau}\right)t} + F_{a,b}^{\frac{\pi}{2}}\left(\frac{1}{\tau}\right)e^{2i\pi\left(\frac{1}{\tau}\right)t}\right\} \tag{4.13}$$

$$\sum_{k=-\infty}^{\infty} f(t+k\tau) = \frac{1}{\tau}\left\{F_{a,b}^{\frac{\pi}{2}}(0) + \sum_{k=1}^{n}\left[F_{a,b}^{\frac{\pi}{2}}\left(\frac{-k}{\tau}\right)e^{-2i\pi\left(\frac{k}{\tau}\right)t} + F_{a,b}^{\frac{\pi}{2}}\left(\frac{k}{\tau}\right)e^{2i\pi\left(\frac{k}{\tau}\right)t}\right]\right\} \tag{4.14}$$

Property 1: Suppose a function $f(t)$ is band-limited to Ω_α in the fractional Fourier domain of order α and

$$h(t) = \sum_{k=-\infty}^{\infty} f(t+k\tau)e^{i\pi a^2\left(2k\tau t + k^2\tau^2\right)\cot\alpha}$$

Then $h(t)$ is a chirp-periodic function with period τ.

Proof: By the definition of chirp-periodicity, we obtain,

$$\begin{aligned} h(t+\tau)e^{i\pi a^2(t+\tau)^2\cot\alpha} &= \sum_{k=-\infty}^{\infty} f(t+\tau+k\tau)e^{i\pi a^2\left(2k\tau(t+\tau)+k^2\tau^2\right)\cot\alpha}e^{i\pi a^2(t+\tau)^2\cot\alpha} \\ &= \sum_{k=-\infty}^{\infty} f(t+(k+1)\tau)e^{i\pi a^2\left(2(k+1)\tau t+(k+1)^2\tau^2\right)\cot\alpha}e^{i\pi a^2 t^2\cot\alpha} \end{aligned}$$

Therefore

$$h(t+\tau)e^{i\pi a^2(t+\tau)^2\cot\alpha} = h(t)e^{i\pi a^2 t^2\cot\alpha}$$

CONCLUSION

In this paper we have introduced the generalized Poisson sum formula based on the relationship of the extended fractional Fourier transform and the Fourier transform and we derived some new results associated with the generalized Poisson sum formula on the extended fractional Fourier transform domain. The special cases are tallies with the (19) and corollary 2 of [2].

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