ORIGINAL PAPER

APPLICATION OF SUMUDU DECOMPOSITION METHOD TO GOURSAT PROBLEM

JAMSHAD AHMAD[1], S. RUBAB[1], M. NAEEM[2], L. A. TAHIR[3]

Manuscript received: 20.11.2014; Accepted paper: 10.12.2014;

Published online: 15.12.2014.

Abstract. *In this paper, we apply Sumudu Decomposition Method for solving Goursat problems which play very important part in applied and engineering sciences. In addition we apply He's polynomials for non linear terms. Several examples are solved to verify the reliability and efficiency of the method.*

Keywords: *Sumudu Decomposition method; He's polynomials; nonlinear problems; Goursat problems*

1. INTRODUCTION

The Goursat partial differential equation arises in linear and non linear partial differential equations with mixed derivatives. There are many approaches that have been suggested to approximate the solution of the Goursat partial differential equation. This paper is devoted to the study of Goursat problems which are known to arise in a variety of physical phenomenon and applied sciences [1-5]. Several techniques including Runge-Kutta, decomposition, finite difference, finite element, geometric mean averaging of the functional values and variational iteration have been used to investigate these problems, see [1-16]. In a later work Ghorbani *et al*. [6,9] split the nonlinear term into a series of polynomials calling them as the He's polynomials. The Sumudu transform can be effectively used to solve ordinary differential equations as well as partial differential equations and engineering problems. In this paper, the main objective is to introduce a comparative study of linear and nonlinear Goursat problems by using (SDM).

In this paper we outline a reliable strategy of the Sumudu Decomposition Method (SDM) for solving the Goursat Problems. The Goursat problem arises in partial differential equation with mixed derivatives. The standard form of Goursat problem [5] is

$$\begin{aligned} &u_{xy} = f(x, y, u, u_x, u_y), \\ &u(x,0) = g(x), u(0, y) = h(y), \\ &g(0) = h(0) = u(0,0), \\ &0 \le x \le a,\ 0 \le y \le b. \end{aligned} \tag{1}$$

[1] Department of Mathematics, Faculty of Sciences, UOG Gujrat-Pakistan. E-mail: Jamshaid_uetian@yahoo.com.
[2] Department of Mathematics, Faculty of Sciences, UET Lahore-Pakistan.
[3] Department of Mathematics, UET RACHNA Campus Gujranwala-Pakistan.

2. SUMUDU DECOMPOSITION METHOD

Over the set of functions

$$A = \{ f(t) : \tau_1, \tau_2 > 0, \left| f(t) \right| < Me^{t/\tau_j}, if\ t \in (-1)^j \times [0,\infty) \}, \tag{2}$$

The Sumudu Transform is defined by

$$G(u) = S[f(t);u] := \int_0^\infty f(ut)e^{-t}dt, \qquad u \in (-\tau_1, \tau_2), \tag{3}$$

Theorem 2.1. Let $f(t)$ be in A, and $F^n(u)$ denotes the Sumudu transform of the n[th] derivative, $f^n(t)$ of, $f(t)$ then for $n > 1$,

$$S_u[f^{(n)}(t)] = F^n(u) = \frac{F(u)}{u^n} - \sum_{j=0}^{n-1} \frac{F^j(0)}{u^{n-j}}, \tag{4}$$

Theorem 2.2. The Sumudu transform amplifies the coefficients of the power series function,

$$f(t) = \sum_{n=0}^\infty a_n t^n, \tag{5}$$

by sending it to the power series function,

$$G(u) = \sum_{n=0}^\infty n! a_n u^n, \tag{6}$$

Derivatives property of Sumudu transform is

$$S[\frac{d^nU}{dx^n}] = \frac{1}{u^n} S[U(x)] - \frac{1}{u^n} U(0) - \frac{1}{u^{n-1}} U'(0) - ... - \frac{U^{n-1}(0)}{u}. \tag{7}$$

3. METHOD OF ANALYSIS

To mention the basic idea of this method, we consider a general nonlinear non-homogeneous Goursat Problem of the form

$$\begin{aligned} &DU(x,t) + RU(x,t) + NU(x,t) = g(x,t), \\ &U(x,0) = h(x),\ U(0,t) = f(t), \end{aligned} \tag{8}$$

where D is the second order linear mixed differential operator $D = \frac{\partial^2}{\partial x \partial t}$, N represent the general nonlinear operator and $g(x,t)$ is the source term.

Taking the Sumudu transform of both sides of Eq. (8), we get

$$S\left[DU(x,t)\right] + S\left[RU(x,t)\right] + S\left[NU(x,t)\right] = S\left[g(x,t)\right], \tag{9}$$

Using the differentiation property of the Sumudu transform and given initial conditions, we have

$$S\left[U_x(x,t)\right] = h(x) - u\,S\left[RU(x,t) + NU(x,t)\right] + u\,S\left[g(x,t)\right], \tag{10}$$

Now, applying the inverse Sumudu transform to both sides of Eq. (10), we get

$$U_x(x,t) = G(x,t) + h(x) - S^{-1}\left[u\,S\left[RU(x,t) + NU(x,t)\right]\right], \tag{11}$$

Again Taking the Sumudu transform of both sides of Eq. (11), we get

$$S[U(x,t)] = f(t) + uS[G(x,t)] + uS[h(x)] - uS\left[S^{-1}\left[uS\left[RU(x,t) + NU(x,t)\right]\right]\right], \tag{12}$$

Now, again applying the inverse Sumudu transform to both sides of Eq. (12), we get

$$U(x,t) = H(x,t) - S^{-1}\left[u\,S\left[S^{-1}\left[u\,S\left[RU(x,t) + NU(x,t)\right]\right]\right]\right], \tag{13}$$

where $H(x,t)$ represents the term arising from the source term and the prescribed initial conditions. We represent solution as an infinite series:

$$u(x,t) = \sum_{i=0}^{\infty} p^i u_i(x,t) \tag{14}$$

And the nonlinear term can be decomposed as

$$NU(x,t) = \sum_{i=0}^{\infty} H_i, \tag{15}$$

where H_i are He's Polynomials of U_0, U_1,...U_n and it can be calculated by formula

$$H_i = \frac{1}{i!}\frac{d^i}{dp^i}[N\sum_{i=0}^{\infty} p^i u_i]_{p=0}, \quad i = 0,1,2,... \tag{16}$$

Then Eq. (13) becomes

$$\sum_{i=0}^{\infty} p^i u_i(x,t) = H(x,t) - S^{-1}\left[u\,S\left[S^{-1}\left[u\,S\left[RU(x,t) + \sum_{i=0}^{\infty} H_i\right]\right]\right]\right], \tag{17}$$

The recursive relation is given by

$$
\begin{aligned}
&p^{0};U_{0}(x,t)=H(x,t),\\
&p^{n+1};U_{n+1}(x,t)=-S^{-1}\left[uS\left[S^{-1}\left[uS\left[RU(x,t)+\sum_{i=0}^{\infty}H_{i}\right]\right]\right]\right],n\geq 0,
\end{aligned}
\tag{18}
$$

4. NUMERICAL APPLICATIONS

Example 4.1 Consider the following homogenous Goursat problem [4]

$$
\begin{aligned}
&u_{xt}=u,\\
&u(x,0)=e^{x},\ u(0,t)=e^{t},u(0,0)=1,
\end{aligned}
\tag{19}
$$

Taking the Sumudu transform of Equation (19) with respect to “t”, we get

$$
\begin{aligned}
&\frac{1}{u}S[U_{x}(x,t)]-\frac{1}{u}U(x,0)=S[u],\\
&S[U_{x}(x,t)]=e^{x}+uS[u],
\end{aligned}
\tag{20}
$$

Taking inverse Sumudu transform of Equation (20), we get

$$U_{x}(x,t)=e^{x}+S^{-1}\left[uS[u]\right], \tag{21}$$

Now, taking again Sumudu transform of Equation (21) with respect to “x”, we get

$$
\begin{aligned}
&\frac{1}{u}S\left[U(x,t)\right]-\frac{1}{u}U(0,t)=S[e^{x}]+S\left[S^{-1}\left[uS[u]\right]\right],\\
&S\left[U(x,t)\right]=e^{t}+\frac{u}{1-u}+uS\left[S^{-1}\left[uS[u]\right]\right],
\end{aligned}
\tag{22}
$$

Taking inverse Sumudu transform of Equation (22), we get

$$U(x,t)=e^{t}-1+e^{x}+S^{-1}\left[uS\left[S^{-1}\left[uS[u]\right]\right]\right], \tag{23}$$

We represent solution as an infinite series:

$$u(x,t)=\sum_{i=0}^{\infty}u_{i}(x,t) \tag{24}$$

then

$$\sum_{i=0}^{\infty}u_{i}(x,t)=e^{t}-1+e^{x}+S^{-1}\left[uS\left[S^{-1}\left[uS[u]\right]\right]\right], \tag{25}$$

The recursive relation is given by

$$u_0(x,t) = e^t - 1 + e^x,$$
$$u_{i+1}(x,t) = S^{-1}\left[uS\left[S^{-1}\left[uS[\sum_{i=0}^{\infty} u_i]\right]\right]\right], i = 0,1,..., \tag{26}$$

Consequently, following approximation obtained,

$$u_0(x,t) = e^t - 1 + e^x,$$
$$u_1(x,t) = -x + xe^t - xt - t + te^x,$$
$$u_2(x,t) = -\frac{x^2}{2!}t - \frac{x^2}{2!} + \frac{x^2}{2!}e^t - \frac{x^2}{2!}\frac{t^2}{2!} - x\frac{t^2}{2!} - \frac{t^2}{2!} + \frac{t^2}{2!}e^x, \tag{27}$$
$$\vdots$$

The closed form solution is given by

$$u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + . \; .$$
$$u(x,t) = e^{x+t}. \tag{28}$$

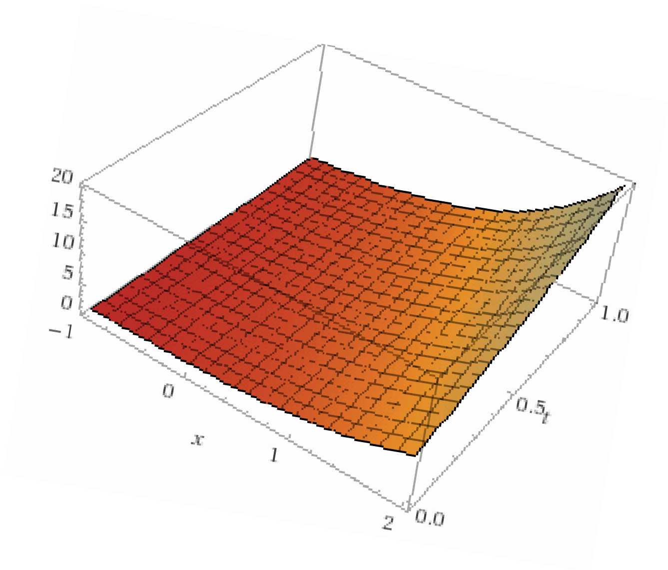

Fig.1. Graphical representation of u(x,t)

Example 4.2 Consider the following homogenous Goursat problem [4]

$$u_{xt} = 2u,$$
$$u(x,0) = e^x,\ u(0,t) = e^{-2t}, u(0,0) = 1, \tag{29}$$

Taking the Sumudu transform of Equation (29) with respect to "*t*", we get

$$\frac{1}{u}S[U_x(x,t)] - \frac{1}{u}U(x,0) = S[2u],$$
$$S[U_x(x,t)] = e^x + uS[2u], \tag{30}$$

Taking inverse Sumudu transform of Equation (30), we get

$$U_x(x,t)=e^x+S^{-1}\left[uS[2u]\right], \tag{31}$$

Now, taking again Sumudu transform of Equation (31) with respect to "*x*", we get

$$\begin{aligned}&\frac{1}{u}S\left[U(x,t)\right]-\frac{1}{u}U(0,t)=S[e^x]+S\left[S^{-1}\left[uS[2u]\right]\right],\\&S\left[U(x,t)\right]=e^{-2t}+\frac{u}{1-u}+uS\left[S^{-1}\left[uS[2u]\right]\right],\end{aligned} \tag{32}$$

Taking inverse Sumudu transform of Equation (32), we get

$$U(x,t)=e^{-2t}-1+e^x+S^{-1}\left[uS\left[S^{-1}\left[uS[2u]\right]\right]\right], \tag{33}$$

We represent solution as an infinite series:

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t) \tag{34}$$

then

$$\sum_{i=0}^{\infty}u_i(x,t)=e^{-2t}-1+e^x+S^{-1}\left[uS\left[S^{-1}\left[uS[2u]\right]\right]\right], \tag{35}$$

The recursive relation is given by

$$\begin{aligned}&u_0(x,t)=e^{-2t}-1+e^x,\\&u_{i+1}(x,t)=S^{-1}\left[uS\left[S^{-1}\left[uS[\sum_{i=0}^{\infty}2u_i]\right]\right]\right], i=0,1,...,\end{aligned} \tag{36}$$

Consequently, following approximation obtained,

$$\begin{aligned}&u_0(x,t)=e^{-2t}-1+e^x,\\&u_1(x,t)=x-xe^{-2t}-2xt-2t+2te^x,\\&u_2(x,t)=\frac{2x^2}{2!}t-\frac{x^2}{2!}+\frac{x^2}{2!}e^{-2t}-\frac{4x^2}{2!}\frac{t^2}{2!}-4x\frac{t^2}{2!}-\frac{4t^2}{2!}+\frac{4t^2}{2!}e^x,\\&\vdots\end{aligned} \tag{37}$$

The closed form solution is given by

$$\begin{aligned}&u(x,t)=u_0(x,t)+u_1(x,t)+u_2(x,t)+. \ .\\&u(x,t)=e^{x-2t}.\end{aligned} \tag{38}$$

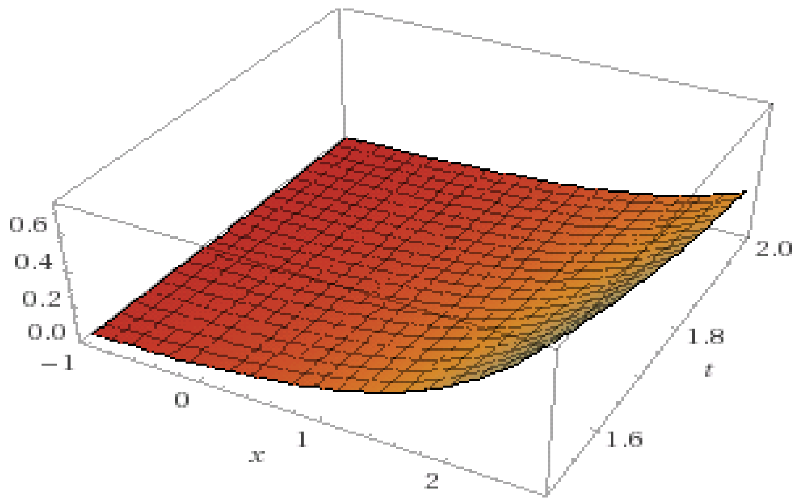

Fig.2. Graphical representation of u(x,t)

Example 4.3 Consider the following inhomogeneous linear Goursat problem [4]

$$u_{xt} = u - t, \\ u(x,0) = e^{x},\ u(0,t) = t + e^{t}, u(0,0) = 1, \tag{39}$$

Taking the Sumudu transform of Equation (39) with respect to "t", we get

$$\frac{1}{u}S[U_x(x,t)] - \frac{1}{u}U(x,0) = S[u-t], \\ S[U_x(x,t)] = e^{x} + uS[u] - u, \tag{40}$$

Taking inverse Sumudu transform of Equation (40), we get

$$U_x(x,t) = e^{x} - t + S^{-1}\left[uS[u]\right], \tag{41}$$

Now, taking again Sumudu transform of Equation (41) with respect to "x", we get

$$\frac{1}{u}S\left[U(x,t)\right] - \frac{1}{u}U(0,t) = S[e^{x} - t] + S\left[S^{-1}\left[uS[u]\right]\right], \\ S\left[U(x,t)\right] = t + e^{t} + \frac{u}{1-u} - ut + uS\left[S^{-1}\left[uS[u]\right]\right], \tag{42}$$

Taking inverse Sumudu transform of Equation (42), we get

$$U(x,t) = t + e^{t} - 1 + e^{x} - xt + S^{-1}\left[uS\left[S^{-1}\left[uS[u]\right]\right]\right], \tag{43}$$

We represent solution as an infinite series:

$$u(x,t) = \sum_{i=0}^{\infty} u_i(x,t) \tag{44}$$

then

$$\sum_{i=0}^{\infty} u_i(x,t) = t + e^t - 1 + e^x - xt + S^{-1}\left[uS\left[S^{-1}\left[uS[u]\right]\right]\right], \tag{45}$$

The recursive relation is given by

$$\begin{aligned} u_0(x,t) &= t + e^t - 1 + e^x - xt, \\ u_{i+1}(x,t) &= S^{-1}\left[uS\left[S^{-1}\left[uS[\sum_{i=0}^{\infty} u_i]\right]\right]\right], i = 0,1,..., \end{aligned} \tag{46}$$

Consequently, following approximation obtained,

$$\begin{aligned} u_0(x,t) &= t + e^t - 1 + e^x - xt, \\ u_1(x,t) &= \frac{xt^2}{2!} - x + xe^t - xt - t + te^x - \frac{x^2}{2!}\frac{t^2}{2!}, \\ u_2(x,t) &= \frac{x^2}{2!}\frac{t^3}{3!} - t\frac{x^2}{2!} - \frac{x^2}{2!} + \frac{x^2}{2!}e^t - \frac{x^2}{2!}\frac{t^2}{2!} - x\frac{t^2}{2!} - \frac{t^2}{2!} + \frac{t^2}{2!}e^x - \frac{x^3}{3!}\frac{t^3}{3!}, \\ &\vdots \end{aligned} \tag{47}$$

The closed form solution is given by

$$\begin{aligned} u(x,t) &= u_0(x,t) + u_1(x,t) + u_2(x,t) + .\ . \\ u(x,t) &= t + e^{x+t}. \end{aligned} \tag{48}$$

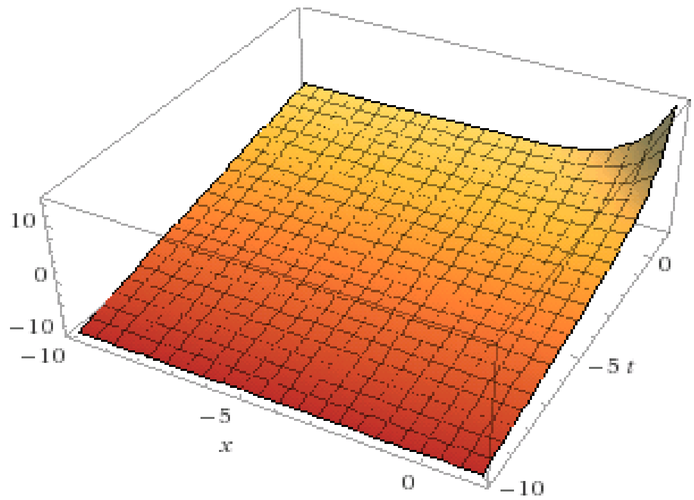

Fig.3. Graphical representation of u(x,t)

Example 4.4 Consider the following inhomogeneous linear Goursat problem [4]

$$\begin{aligned} u_{xt} &= u + 4xt - x^2t^2, \\ u(x,0) &= e^x,\ u(0,t) = e^t, u(0,0) = 1, \end{aligned} \tag{49}$$

Taking the Sumudu transform of Equation (49) with respect to "t", we get

$$\frac{1}{u}S[U_x(x,t)]-\frac{1}{u}U(x,0)=S[u+4xt-x^2t^2],$$
$$S[U_x(x,t)]=e^x+4xu^2-2x^2u^3+uS[u], \tag{50}$$

Taking inverse Sumudu transform of Equation (50), we get

$$U_x(x,t)=e^x+\frac{4xt^2}{2!}-2x^2\frac{t^3}{3!}+S^{-1}\left[uS[u]\right], \tag{51}$$

taking again Sumudu transform of Equation (51) with respect to “x”, we get

$$\frac{1}{u}S\left[U(x,t)\right]-\frac{1}{u}U(0,t)=S[e^x+\frac{4xt^2}{2!}-2x^2\frac{t^3}{3!}]+S\left[S^{-1}\left[uS[u]\right]\right],$$
$$S\left[U(x,t)\right]=e^t+\frac{u}{1-u}+4u^2\frac{t^2}{2!}-4u^3\frac{t^3}{3!}+uS\left[S^{-1}\left[uS[u]\right]\right], \tag{52}$$

Taking inverse Sumudu transform of Equation (52), we get

$$U(x,t)=e^t-1+e^x+4\frac{x^2}{2!}\frac{t^2}{2!}-4\frac{x^3}{3!}\frac{t^3}{3!}+S^{-1}\left[uS\left[S^{-1}\left[uS[u]\right]\right]\right], \tag{53}$$

We represent solution as an infinite series:

$$u(x,t)=\sum_{i=0}^{\infty}u_i(x,t) \tag{54}$$

then

$$\sum_{i=0}^{\infty}u_i(x,t)=e^t-1+e^x+4\frac{x^2}{2!}\frac{t^2}{2!}-4\frac{x^3}{3!}\frac{t^3}{3!}+S^{-1}\left[uS\left[S^{-1}\left[uS[u]\right]\right]\right], \tag{55}$$

The recursive relation is given by

$$u_0(x,t)=e^t-1+e^x+4\frac{x^2}{2!}\frac{t^2}{2!}-4\frac{x^3}{3!}\frac{t^3}{3!},$$
$$u_{i+1}(x,t)=S^{-1}\left[uS\left[S^{-1}\left[uS[\sum_{i=0}^{\infty}u_i]\right]\right]\right], i=0,1,..., \tag{56}$$

Consequently, following approximation obtained,

$$
\begin{aligned}
&u_0(x,t) = e^t - 1 + e^x + 4\frac{x^2}{2!}\frac{t^2}{2!} - 4\frac{x^3}{3!}\frac{t^3}{3!},\\
&u_1(x,t) = -x + xe^t - xt - t + te^x + 4\frac{x^3}{3!}\frac{t^3}{3!} - 4\frac{x^4}{4!}\frac{t^4}{4!},\\
&u_2(x,t) = -t\frac{x^2}{2!} - \frac{x^2}{2!} + \frac{x^2}{2!}e^t - \frac{x^2}{2!}\frac{t^2}{2!} - x\frac{t^2}{2!} - \frac{t^2}{2!} + \frac{t^2}{2!}e^x + 4\frac{x^4}{4!}\frac{t^4}{4!} - 4\frac{x^5}{5!}\frac{t^5}{5!},\\
&\vdots
\end{aligned} \tag{57}
$$

The closed form solution is given by

$$
\begin{aligned}
&u(x,t) = u_0(x,t) + u_1(x,t) + u_2(x,t) + \ldots\\
&u(x,t) = x^2t^2 + e^{x+t}.
\end{aligned} \tag{58}
$$

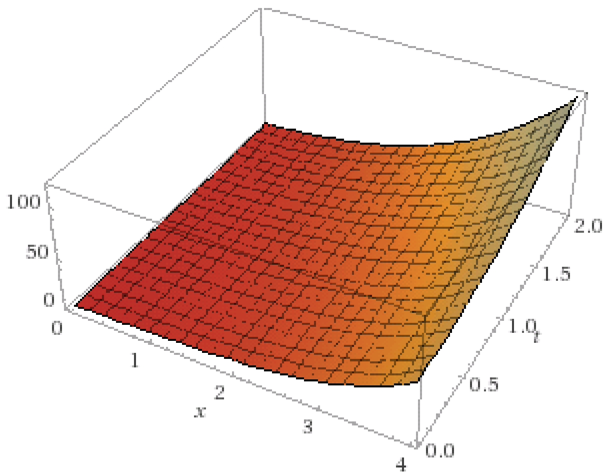

Fig.3. Graphical representation of u(x,t)

Example 4.5 Consider the following nonlinear inhomogeneous Goursat problem [4]

$$
\begin{aligned}
&u_{xt} = -u^2 + e^{2x} + te^{2t} + 2e^{x+t},\\
&u(x,0) = 1 + e^x,\ u(0,t) = 1 + e^t, u(0,0) = 2,
\end{aligned} \tag{59}
$$

Taking the Sumudu transform of Equation (59) with respect to "t", we get

$$
\begin{aligned}
&\frac{1}{u}S[U_x(x,t)] - \frac{1}{u}U(x,0) = S[-u^2 + e^{2x} + te^{2t} + 2e^{x+t}],\\
&S[U_x(x,t)] = 1 + e^x + ue^{2x} + \frac{2ue^x}{1-u} + \frac{u^2}{(1-2u)^2} - uS[u^2],
\end{aligned} \tag{60}
$$

Taking inverse Sumudu transform of Equation (60), we get

$$
U_x(x,t) = 1 - e^x + te^{2x} + 2e^{x+t} + \frac{t^2}{2!}e^{2t} - S^{-1}\left[uS[u^2]\right], \tag{61}
$$

Now, taking again Sumudu transform of Equation (61) with respect to "x", we get

$$\begin{aligned}&\frac{1}{u}S\left[U(x,t)\right]-\frac{1}{u}U(0,t)=S[1-e^{x}+te^{2x}+2e^{x+t}+\frac{t^{2}}{2!}e^{2t}]-S\left[S^{-1}\left[uS[u^{2}]\right]\right],\\&S\left[U(x,t)\right]=1+e^{t}+u-\frac{u}{1-u}+\frac{ut}{(1-2u)}+\frac{2ue^{t}}{1-u}+\frac{ut^{2}}{2!}e^{2t}-uS\left[S^{-1}\left[uS[u^{2}]\right]\right],\end{aligned} \tag{62}$$

Taking inverse Sumudu transform of Equation (62), we get

$$U(x,t)=-e^{t}-e^{x}+x-\frac{t}{2}+\frac{t}{2}e^{x}+2e^{x+t}+x\frac{t^{2}}{2!}e^{2t}-S^{-1}\left[uS\left[S^{-1}\left[uS[u^{2}]\right]\right]\right], \tag{63}$$

We represent solution as an infinite series:

$$u(x,t)=\sum_{i=0}^{\infty}u_{i}(x,t) \tag{64}$$

and the nonlinear term can be decomposed as

$$u^{2}=\sum_{i=0}^{\infty}H_{i}, \tag{65}$$

where H_i are He's Polynomials of U_0, $U_1,...U_n$ and it can be calculated by formula

$$H_{i}=\frac{1}{i!}\frac{d^{i}}{dp^{i}}[N\sum_{i=0}^{\infty}p^{i}u_{i}]_{p=0},\qquad i=0,1,2,... \tag{66}$$

then

$$\sum_{i=0}^{\infty}u_{i}(x,t)=-e^{t}-e^{x}+x-\frac{t}{2}+\frac{t}{2}e^{x}+2e^{x+t}+x\frac{t^{2}}{2!}e^{2t}-S^{-1}\left[uS\left[S^{-1}\left[uS[\sum_{i=0}^{\infty}H_{i}]\right]\right]\right], \tag{67}$$

Comparing the coefficients of "p", we get

$$\begin{aligned}&p^{0};u_{0}(x,t)=-e^{t}-e^{x}+x-\frac{t}{2}+\frac{t}{2}e^{x}+2e^{x+t}+x\frac{t^{2}}{2!}e^{2t},\\&p^{i+1};u_{i+1}(x,t)=-S^{-1}\left[uS\left[S^{-1}\left[uS[\sum_{i=0}^{\infty}H_{i}]\right]\right]\right],i=0,1,...,\end{aligned} \tag{68}$$

The closed form solution is given by

$$\begin{aligned}&u(x,t)=u_{0}(x,t)+u_{1}(x,t)+u_{2}(x,t)+.\ .\\&u(x,t)=e^{t}+e^{x}.\end{aligned} \tag{69}$$

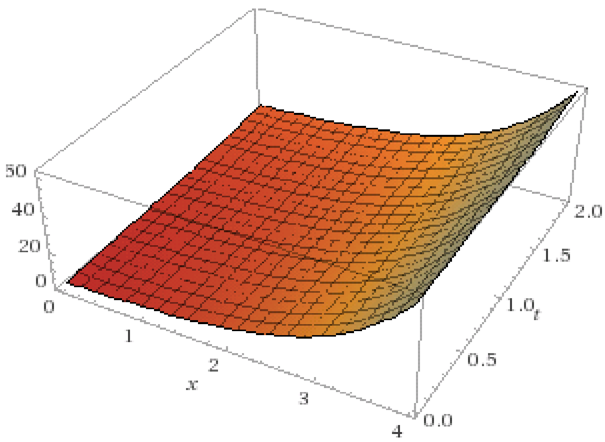

Fig.5. Graphical representation of u(x,t)

5. CONCLUSION

In this paper, we applied the Sumudu Decomposition Method (SDM) for solving Goursat problems. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. The method gives more realistic series solutions that converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result.

REFERENCES

[1] Day, J. T., *Comput.T.*, **9**, 81, 1966.
[2] Evans, D. J., Sangui, B. B., *Appl. Math. Lett.*, **1**, 221, 1988.
[3] Wazwaz, A. M., *Appl. Math. Comput.*, **59**, 89, 1993.
[4] Ahmad, J., Mohyud-Din, S. T., *Life Sci J*; **10**(4), 210, 2013.
[5] Wazwaz, A. M., *Appl. Math. Comput.*, **69**, 299, 1995.
[6] Wazwaz, A. M., *Appl. Math. Comput.*, **193**, 455, 2007.
[7] Ahmad, J., Bibi, Z., Noor, K., *Journal of Science and Arts*, **2**(27), 131, 2014.
[8] Ghorbani, A., Nadjfi, J. S., *Int. J. Nonlin. Scne. Num. Simul.*, **8**(2), 229, 2007.
[9] Ghorbani, A., *Chaos, Solitons and Fractals*, **39**(3), 1486, 2009.
[10] Kilicman, A., Eltayeb, H., *Applied Mathematical Sciences*, **4**(3), 109, 2010.
[11] Asiru, M. A., *Int. J. of Math. Education in Sci.&Techn.*, **32**(6), 906, 2001.
[12] Asiru, M. A., *Int. J. of Math. Education in Sci.&Techn.*, **33**(3), 441, 2002.
[13] Eltayeb, H., Kilicman, A., Fisher, B., *Integral Transforms and Special Functions*, **21**(5), 367, 2010.
[14] Kilicman, A., Eltayeb, H., *Journal of the Franklin Institute*, **347**(5), 848, 2010.
[15] Kilicman, A., Eltayeb, H., Agarwal, R. P., *Abstract and Applied Analysis*, **2010**, Article ID 598702, 2010.
[16] Aseeri, S. A., *Int. J.for Open Problems Computer Mathematic*, **1**, 266, 2008.