ORIGINAL PAPER

α-LIMIT SETS OF SUBSETS OF A METRIC SPACE

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Abstract. *In this paper we prove some properties of limit sets of subsets in a metric space. We also prove that if the limit set of a subset is compact then it should be connected.*

Keywords: *metric space, α-limit, subsets.*

1. INTRODUCTION

The concept of α-limit set of a subset Y in a metric space (X , d) on which there is a flow $f : X \times \mathbb{R} \to X$, is introduced by Changming Ding in [2]. Limit sets play an important role in understanding the dynamics of a system. Various types of limit sets are defined and studied by many authors. In this paper we made an attempt to study some properties of α-limit sets of subsets of a metric space. We also proved that if the α-limit set of a subset is compact then it should be connected.

2. PRELIMINARIES

Throughout this paper X is a compact metric space. A flow on X is a mapping $f : X \times \mathbb{R} \to X$ such that for all $x \in X$ and real numbers s and t, $f(x,0) = x$ and $f(f(x,t),s) = f(x,s+t)$. For a set $Y \subseteq X$ and $J \subseteq \mathbb{R}$, we define $Y \cdot J = \{x \cdot t = f(x,t) : x \in Y, t \in J\}$. A subset Y of X is invariant under f if $Y \cdot \mathbb{R} = Y$. The α-limit set of a subset Y of X is defined as $\alpha(Y) = \bigcap_{t \geq 0} \overline{Y \cdot (-\infty, -t]}$ (see [2]).

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3. RESULTS

Lemma 1. For any subset Y of X, $\alpha(Y)=\bigcap_{n\geq 0}\overline{Y\cdot(-\infty,-n]}$.

Proof: If $x\in\alpha(Y)$, then $x\in\overline{Y\cdot(-\infty,-t]}$ for each $t\geq 0$. So, $x\in\overline{Y\cdot(-\infty,-n]}$ for each $n\geq 0$ and hence $x\in\bigcap_{n\geq 0}\overline{Y\cdot(-\infty,-n]}$. For the other part, we note that $\overline{Y\cdot(-\infty,-n]}=\bigcap_{n-1<t\leq n} Y\cdot(-\infty,-t]$. From this we get that $\bigcap_{n\geq 0}\overline{Y\cdot(-\infty,-n)}\subseteq\bigcap_{t\geq 0}\overline{Y\cdot(-\infty,-t]}$.

Lemma 2. If Y and Z are subsets of X such that $Y\subset Z$ then $\alpha(Y)\subset\alpha(Z)$.

Proof. If $x\in\alpha(Y)$ then $x\in\overline{Y\cdot(-\infty,-n]}$ for each $n\geq 0$. Since $Y\subset Z$, $x\in\overline{Z\cdot(-\infty,-n]}$ for each $n\geq 0$. This shows that $x\in\bigcap_{n\geq 0}\overline{Z\cdot(-\infty,-n]}$, which gives $\alpha(Y)\subset\alpha(Z)$.

Remark 1. For any subset Y of X, $\alpha(Y)\subseteq\alpha(\overline{Y})$.

Lemma 3. If A and B are any two subsets of X, then $\alpha(A\cup B)=\alpha(A)\cup\alpha(B)$.

Proof: We have $A\subseteq(A\cup B)$ and $B\subseteq(A\cup B)$ implying $\alpha(A)\cup\alpha(B)\subseteq\alpha(A\cup B)$.
On the other hand, $\alpha(A\cup B)=\bigcap_{n\geq 0}\overline{(A\cup B)\cdot(-\infty,-n)}=\bigcap_{n\geq 0}\overline{(A\cdot(-\infty,-n])\cup(B\cdot(-\infty,-n])}$
$\subseteq\bigcap_{n\geq 0}\overline{(A\cdot(-\infty,-n]}\cup\overline{(B\cdot(-\infty,-n])}\subseteq\bigcap_{n\geq 0}\overline{A\cdot(-\infty,-n]}\cup\bigcap_{n\geq 0}\overline{B\cdot(-\infty,-n]}=\alpha(A)\cup\alpha(B)$.

Lemma 4. For any subset Y of X, $z\in\alpha(Y)\Leftrightarrow$ there are sequences $y_n\in Y$ and $t_n\in\mathbb{R}$, $t_n\to-\infty$ as $n\to\infty$ such that $\lim_{n\to\infty} y_n\cdot t_n=z$.

Proof: Let $z\in\alpha(Y)$ then $z\in\overline{Y\cdot(-\infty,-n]}$ for each $n\geq 0$ and so there exists sequences $y_n\in Y$ and $t_n\in(-\infty,-n]$ such that $\lim_{n\to\infty} y_n\cdot t_n=z$. As $t_n\in(-\infty,-n]$, we have $t_n\leq -n$ which implies $\lim_{n\to\infty} t_n\leq\lim_{n\to\infty}(-n)=-\infty$ so that $t_n\to-\infty$ as $n\to\infty$.

Conversely, suppose that there is a point z in X and there are two sequences $y_n\in Y$ and $t_n\in\mathbb{R}$, $t_n\to-\infty$ as $n\to\infty$ such that $\lim_{n\to\infty} y_n\cdot t_n=z$. Since $t_n\to-\infty$ as $n\to\infty$, we get

$t_n \le -n$, which means $t_n \in (-\infty, -n]$. As $y_n \in Y$ and $t_n \in (-\infty, -n]$, we get $y_n \cdot t_n \in Y \cdot (-\infty, -n]$. Since $\lim_{n\to\infty} y_n \cdot t_n = z$, we get that $z \in \alpha(Y)$. This proves the Lemma.

Lemma 5. For any subset Y of X, $\alpha(Y)$ is non-empty, closed and invariant.

Proof: By definition, the set $\alpha(Y)$ is the intersection of closed sets , so it is closed. Since these sets are nested, by the compactness of X we get that $\alpha(Y)$ is nonempty. It remains to prove the invariance. Let $\alpha(Y) = A$. We prove that $A \cdot \mathbb{R} = A$. By definition of $A \cdot \mathbb{R}$ it is clear that $A \subseteq A \cdot \mathbb{R}$.

Suppose $x \in A \cdot \mathbb{R}$. Then , $x = a \cdot r$ wherw $a \in A$ and $r \in \mathbb{R}$. Since $a \in A$, by Lemma 4, there are two sequences $y_n \in Y$ and $t_n \in \mathbb{R}$, $t_n \to -\infty$ as $n \to \infty$ such that $\lim_{n\to\infty} y_n \cdot t_n = a$.
So, $x = a \cdot r = \lim_{n\to\infty} \{y_n \cdot t_n\} \cdot r = \lim_{n\to\infty} y_n \cdot \{t_n \cdot r\} = \lim_{n\to\infty} y_n \cdot p_n$, where $p_n = t_n \cdot r \in \mathbb{R}$.
Now, by Lemma 4, it follows that $x \in A$.

Lemma 6. Suppose X is a Locally Compact metric space. Let $\{A_n\}$ be a decreasing sequence of closed, connected subsets of X uch that $\bigcap_{n=1}^{\infty} A_n$ is a nonempty compact subset of X . Then, for an arbitrary neighborhood U of $\bigcap_{n=1}^{\infty} A_n$, there is a natural number n such that $A_n \subseteq U$.

Proof: Let $A = \bigcap_{n=1}^{\infty} A_n$. Suppose on the contrary there exists a neighborhood U of A such that $A_n \not\subseteq U$ for each natural number n. Since A is the compact subset of a locally compact space X , there exists a neighborhood V of A such that $\overline{V} \subseteq U$ and it is compact. Since for each n, A_n is not contained in U , the set $A_n \cap V^c$ is nonempty. Also $A_n \cap V$ contains the set A, it is nonempty. From the connectedness of the set A_n it intersects to the boundary of V , ∂V . So, we can choose an element x_n in ∂V . Since ∂V is also compact, there exists a convergent subsequence of $\{x_n\}$ in ∂V . Without loss of generality we can assume that the sequence $\{x_n\}$ converges to a point x in ∂V . Let k be an arbitrary natural number. For $n > k$, $x_n \in A_k$. So, the limit point x is an element of $\overline{A_k}$. Since A_n is closed, $x \in A$. Hence , we derive a contradiction from the fact that V is the neighborhood of A. So , we can find a natural number n such that $A_n \subseteq U$.

Theorem 1. *Suppose X is a locally compact metric space and $Y \subseteq X$ is a connected set. If $\alpha(Y)$ is compact then $\alpha(Y)$ is connected.*

Proof: Let us assume that $\alpha(Y)$ is not connected. Then, there exists two disjoint clopen sets A and B of $\alpha(Y)$. Since X is locally compact A and B are also compact in X. Let U and V be the neighborhoods of A and B respectively. Put $A_k = \overline{Y \cdot (-\infty, -n]}$. Then $\{A_k\}$ is a decreasing sequence of closed subsets of X. By Lemma 6 , there exists a natural number n such that $A \subseteq U \bigcup V$. Since A_n is connected, either $A_n \subseteq U$ or $A_n \subseteq V$. Suppose $A_n \subseteq U$. If $k \geq n$ we obtain that $A_k \subseteq A_n \subseteq U$ and so $\alpha(Y) \subseteq U$. This is a contradiction . Thus $\alpha(Y)$ is connected.

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