ORIGINAL PAPER

EXACT SOLUTION OF WHITHAM-BROER-KAUP SHALLOW WATER WAVE EQUATIONS

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Abstract.*In this study we present the analysis of Adomian Decomposition Method using He's Polynomials for nonlinear WhithamBroer-Kaup equations dealing with propagation of shallow water waves with different dispersion relations. The exact solutions of two variants of WhithamBroer-Kaup equations are studied. The suggested method is used without discretization, linearization or restrictive assumptions. Numerical results show that the proposed method was efficient and capable to obtain the exact solution of this set of wave equations. The obtained solutions of these equations could straightforwardly show some facts of the described process deeply such as the propagation. It is clear that this method can be easily extended to other nonlinear wave equations arising in mathematical physics.*

Keywords: *Nonlinear PDEs, Adomian decomposition (ADM) using He's polynomials, Coupled WhithamBroer-Kaup equations.*

1. INTRODUCTION

Nonlinearity exists everywhere and nature is nonlinear in general. Nonlinear physical phenomena that appear in many areas of scientific fields such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics can be modeled by partial differential equation. Most of physical systems can be described by appropriate sets of differential equations, which are well suited as models for systems. Hence, understanding of differential equations and finding its solutions are of primary importance for researchers, mathematicians as for physicists. A broad class of analytical solutions methods and numerical solutions methods were used in handle these mathematical problems [1-16].

In this paper, we consider the coupled Whithamï Broerï Kaup (WBK) equations which have been studied by Whitham [1], Broer [2] and Kaup [3]. The equations describe the propagation of shallow water waves, with different dispersion relations. The WBK equations are as follows,

$$\begin{aligned} &u_t + uu_x + v_x + \beta u_{xx} = 0 \\ &v_t + vu_x + uv_x - \beta v_{xx} + \alpha u_{xxx} = 0 \end{aligned} \tag{1}$$

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where $u = u(x,t)$ is the horizontal velocity, $v = v(x,t)$ is the height that deviates from equilibrium position of the liquid, and α, β are constants which are represented in different diffusion powers [4]. The basic motivation of this work is to apply the Adomian Decomposition Method (ADM) coupled with He's polynomials to find travelling wave solutions of Whitham-Broer-Kaup (WBK) equations which arise quite frequently in mathematical physics, nonlinear sciences.

It is shown that the proposed ADM provides the solution in a rapid convergent series with easily computable components. Numerical results explicitly reveal the complete reliability of the proposed algorithms.

2. ANALYSIS ADOMIAN DECOMPOSITION METHOD USING HE'S POLYNOMIALS

To illustrate the basic concept of He's Adomian decomposition, we consider the following general differential equation

$$L(u) + N(u) = g(x) \tag{2}$$

where L is the linear operator and N is the nonlinear operator and *g(x)* is the homogeneous term.

According to the ADM we construct the

$$u_{n+1} = u_n - L_t^{-1}\{N(u) + g(x)\} \tag{3}$$

where $L_t^{-1} = \int_0^t dt$. The embedding parameter $p \in (0, 1]$ can be considered as an Expanding parameters. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter to obtain

$$u = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots \tag{4}$$

If $p \to 1$,the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{n=0}^{\infty} u_n \tag{5}$$

It is well known that series (3) is convergent for most of the cases and also the rate of convergence is dependent on *L(u)*. We assume that (3) has a unique solution. The comparisons of like powers of p give solutions of various orders. In sum, according to, He's considers the solution $u(x)$, of the homotopy equation in a series of p as follows:

$$u(x) = \sum_{n=0}^{\infty} p^n u_n = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \ldots$$

and the method consider the nonlinear term *N(u)*, as

$$N(u)=\sum_{n=0}^{\infty}p^nH_n=H_0+pH_1+p^2H_2+p^3H_3+\dots$$

where H_n are so called He's polynomials which can be calculated by using the formula

$$H_n(u_0,u_1,u_2,\dots)=\frac{1}{n!}\frac{\partial^n}{\partial p^n}\left[N\left(\sum_{i=0}^{\infty}p^iu_i\right)\right]_{p=0},\qquad n=0,1,2,3,\dots \tag{6}$$

The successive approximation $u_{n+1},\ n\geq 0$ of the solution of u will be obtained by selective function u_0. Consequently the solution is given by $u=\lim_{n\to\infty}u_n$.

3. NUMERICAL APPLICATIONS

In this section, we apply the Adomian decomposition method using He's polynomials for solving coupled Whitham-Broer-Kaupequations.

Example 3.1 Consider the equations (1) with $\alpha=0$ and $\beta=\frac{1}{2}$**,** we have

$$u_t=-uu_x-v_x-\frac{1}{2}u_{xx}$$

$$v_t=-uv_x-vu_x+\frac{1}{2}v_{xx} \tag{7}$$

with subject to initial conditions

$$u(x,0)=\omega-k\coth\left[k(x+x_0)\right]$$

$$v(x,0)=-k^2\operatorname{csch}^2\left[k(x+x_0)\right] \tag{8}$$

According to the above procedure

$$u(x,t)=\omega-k\coth\left[k(x+x_0)\right]-pL_t^{-1}\left\{\sum_{n=0}^{\infty}p^n(u_nu_{nx})+\sum_{n=0}^{\infty}p^n(v_{nx})+\sum_{n=0}^{\infty}p^n\left(\frac{1}{2}u_{nxx}\right)\right\}$$

$$v(x,t)=-k^2\operatorname{csch}^2\left[k(x+x_0)\right]-pL_t^{-1}\left\{\sum_{n=0}^{\infty}p^n(u_nv_{nx})+\sum_{n=0}^{\infty}p^n(vu_{nx})-\sum_{n=0}^{\infty}p^n\left(\frac{1}{2}v_{nxx}\right)\right\} \tag{9}$$

$$u(x,t)=\omega-k\coth\left[k(x+x_0)\right]-pL_t^{-1}\left\{\sum_{n=0}^{\infty}p^n(H_n(u))+\sum_{n=0}^{\infty}p^n(v_{nx})+\sum_{n=0}^{\infty}p^n\left(\frac{1}{2}u_{nxx}\right)\right\}$$

$$v(x,t) = -k^2 \operatorname{csc}h^2\left[k\left(x+x_0\right)\right] - pL_t^{-1}\left\{\sum_{n=0}^{\infty} p^n\left(A_n\right) + \sum_{n=0}^{\infty} p^n\left(B_n\right) - \sum_{n=0}^{\infty} p^n\left(\frac{1}{2}v_{nxx}\right)\right\} \quad (10)$$

where A_n, B_n and H_n are nonlinear terms, comparing like powers components of p, we get

$$p^0 : u_0 = \omega - k\cot h\left[k\left(x+x_0\right)\right]$$

$$p^0 : v_0 = -k^2 \operatorname{csc}h^2\left[k\left(x+x_0\right)\right]$$

$$p^1 : u_1 = -L_t^{-1}\left\{H_0 + v_{0x} + \frac{1}{2}u_{0xx}\right\} = -\omega k^2 \operatorname{csc}h^2\left[k\left(x+x_0\right)\right]t$$

$$p^1 : v_1 = -L_t^{-1}\left\{A_0 + B_0 - \frac{1}{2}v_{0xx}\right\} = -2\omega k^3 \operatorname{csc}h^2\left[k\left(x+x_0\right)\right]\cot h\left[k\left(x+x_0\right)\right]t$$

And so on, summing all components of *u(x,t) and v(x,t)*, we get the series solution

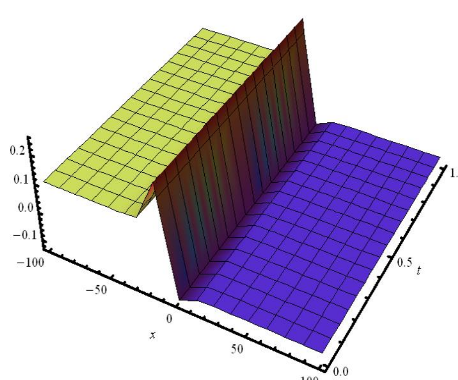
$$u(x,t) = \omega - k\cot h\left[k\left(x+x_0\right)\right] - \omega k^2 \operatorname{csc}h^2\left[k\left(x+x_0\right)\right]t + \ldots$$

$$v(x,t) = -k^2 \operatorname{csc}h^2\left[k\left(x+x_0\right)\right] - 2\omega k^3 \operatorname{csc}h^2\left[k\left(x+x_0\right)\right]\cot h\left[k\left(x+x_0\right)\right]t + \ldots$$

The closed form solution is

$$u(x,t) = \omega - k\cot h\left[k\left(x+x_0-\omega t\right)\right]$$

$$v(x,t) = -k^2 \operatorname{csc}h^2\left[k\left(x+x_0-\omega t\right)\right].$$



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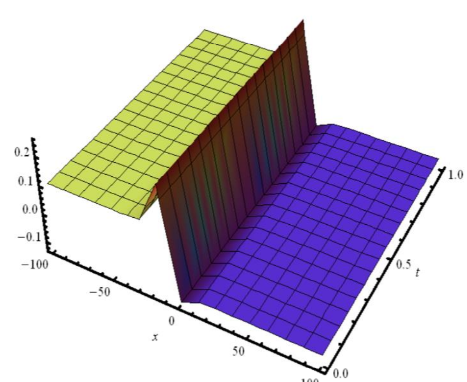
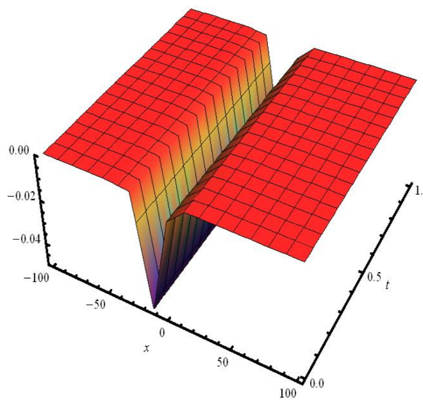


(b)

Fig. 1. (a) Exact solution and (b) Approximate solution of $u(x,t)$ **with** $-100 \le x \le 100$ **and** $0 \le t \le 1$ **and** $\omega = 0.005$, $k = 0.1$, $x_0 = 10$.



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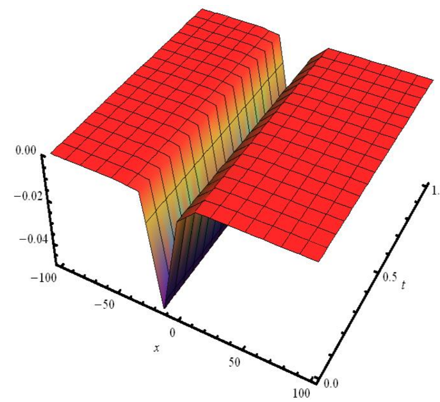


(b)

Fig.2. (a) Exact solution and (b) Approximate solution of $v(x,t)$ with $-100 \le x \le 100$ and $0 \le t \le 1$ and $\omega = 0.005$, $k = 0.1$, $x_0 = 10$.

Example 2.2 Consider the equations (1) with $\alpha = 3$ and $\beta = 1$, we have

$$u_t = -uu_x - v_x - u_{xx},$$

$$v_t = -uv_x - vu_x + v_{xx} - 3u_{xxx}, \tag{11}$$

with subject to initial conditions

$$u(x,0) = \frac{1}{2} - 8\tan h(-2x)$$

$$v(x,0) = 16 - 16\tan h^2(-2x) \tag{12}$$

According to the above procedure

$$u(x,t) = \frac{1}{2} - 8\tan h(-2x) - pL_t^{-1}\left\{\sum_{n=0}^{\infty} p^n(u_n u_{nx}) + \sum_{n=0}^{\infty} p^n(v_{nx}) + \sum_{n=0}^{\infty} p^n(u_{nxx})\right\} \tag{13}$$

$$v(x,t) = 16 - 16\tan h^2(-2x) - pL_t^{-1}\left\{\sum_{n=0}^{\infty} p^n(u_n v_{nx}) + \sum_{n=0}^{\infty} p^n(vu_{nx}) - \sum_{n=0}^{\infty} p^n(v_{nxx}) + 3\sum_{n=0}^{\infty} p^n(u_{nxxx})\right\}$$

$$u(x,t) = \frac{1}{2} - 8\tan h(-2x) - pL_t^{-1}\left\{\sum_{n=0}^{\infty} p^n(H_n(u)) + \sum_{n=0}^{\infty} p^n(v_{nx}) + \sum_{n=0}^{\infty} p^n(u_{nxx})\right\} \tag{14}$$

$$v(x,t) = 16 - 16\tan h^2(-2x) - pL_t^{-1}\left\{\sum_{n=0}^{\infty} p^n(A_n) + \sum_{n=0}^{\infty} p^n(B_n) - \sum_{n=0}^{\infty} p^n(v_{nxx}) + 3\sum_{n=0}^{\infty} p^n(u_{nxxx})\right\}$$

where A_n, B_n and H_n are nonlinear terms, comparing like powers components of p, we get

$$p^0 : u_0 = \frac{1}{2} - 8\tan h(-2x)$$

$$p^0 : v_0 = 16 - 16\tan h^2(-2x)$$

$$p^1 : u_1 = -L_t^{-1}\left\{H_0 + v_{0x} + \frac{1}{2}u_{0xx}\right\} = -8\sec h^2(-2x)t$$

$$p^1 : v_1 = -L_t^{-1}\{A_0 + B_0 - v_{0xx} + 3u_{0xxx}\} = -32\sec h^2(-2x)\tan h(-2x)t$$

And so on, summing all components of *u(x,t) and v(x,t)*, we get the series solution

$$u(x,t) = u_0 + u_1 + ...$$

$$u(x,t) = \frac{1}{2} - 8\tan h(-2x) - 8\sec h^2(-2x)t + ...$$

$$v(x,y) = v_0 + v_1 + ...$$

$$v(x,y) = 16 - 16\tan h^2(-2x) - 32\sec h^2(-2x)\tan h(-2x)t + ...$$

The closed form solution is

$$u(x,t) = \frac{1}{2} - 8\tan h\left[-2\left(x - \frac{1}{2}t\right)\right]$$

$$v(x,y) = 16 - 16\tan h^2\left[-2\left(x - \frac{1}{2}t\right)\right].$$

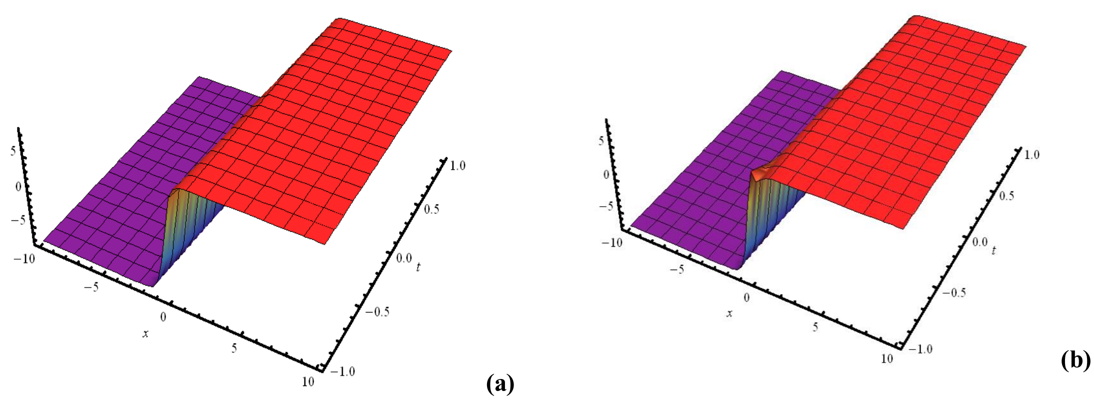
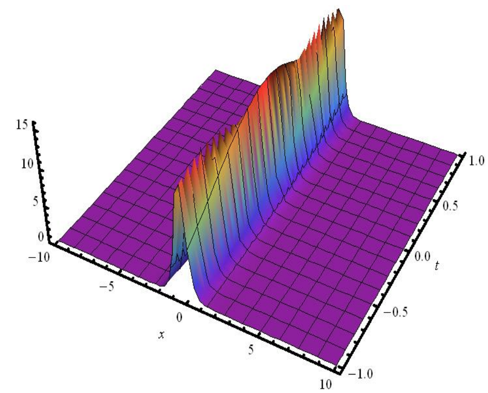
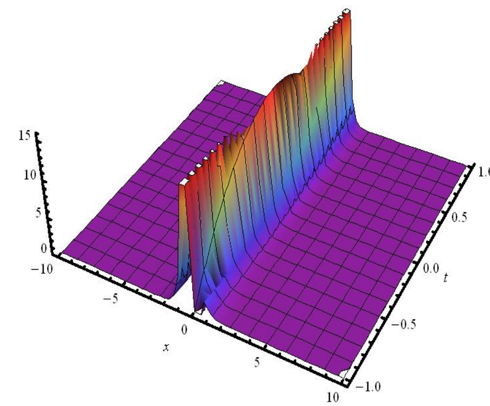


Fig. 3. (a) Exact solution and (b) Approximate solution of $u(x,t)$ with $-10 \le x \le 10$ and $-1 \le t \le 1$.



(a)



(b)

Fig.2. (a) Exact solution and (b) Approximate solution of $v(x,t)$ **with** $-10 \le x \le 10$ **and** $-1 \le t \le 1$**.**

4. CONCLUSIONS

In this study, the Adomian Decomposition method using He’s polynomials has been used for finding the exact and approximate traveling wave solutions of the Whitham–Broer–Kaup (WBK) equations in shallow water. This method is used without in a direct way without using linearization, transformation, discretization or restrictive assumptions. From the obtained results, it may be concluded that the Adomian decomposition method using He’s polynomials is a very powerful and efficient technique to find exact or approximate solutions for a wide classes of problems. It is also worth pointing out that the advantage of ADM is the fast convergence to the solutions.

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