ORIGINAL PAPER

A TWO SIDED ESTIMATE OF $e^x - \left(1+\frac{x}{t}\right)^t$ REVISITED

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Abstract. *In this paper we show that the two sided estimate of* $e^x - (1+x/t)^t$*, stated in [1, Theorem 1], must be reversed for* $t > 0$ *and* $0 < x < 1$*. This modification of Theorem 1 in [1] led us to revisit the short proof given in [1] and improve the condition (*$t > (1-x)/2$*) for an inequality bounding* $e^x - (1+x/t)^t$*, established in [2].*
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1. INTRODUCTION

It has been shown in [1, Theorem 1], by using an elementary consequence of Lagrange Theorem, that for all real numbers $x, t > 0$, the following double inequality holds

$$x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1} < e^x - \left(1+\frac{x}{t}\right)^t < x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)e^{x-1}. \tag{1}$$

As an application, the authors in [1] used (1) together with the estimate

$$\frac{ex}{2t+2x} < e - \left(1+\frac{x}{t}\right)^{\frac{t}{x}} < \frac{ex}{2t+x} \qquad (x, t > 0) \tag{2}$$

in order to give a simple proof of the following double inequality

$$\frac{x^2 e^x}{2t+x+\max(x,x^2)} < e^x - \left(1+\frac{x}{t}\right)^t < \frac{x^2e^x}{2t+x} \qquad \left(x>0, t>\frac{1-x}{2}\right), \tag{3}$$

stated in [2, Theorem 1].

But after some numerical computations, we observe that (1) is not valid for $x < 1$. For instance, taking $x = t = 1/2$ in (1) we obtain

$$\frac{1}{2}(e-2)\frac{1}{\sqrt{2}} < \sqrt{e} - \sqrt{2} < \frac{1}{2}(e-2)\frac{1}{\sqrt{e}}. \tag{4}$$

So, estimate (4) is not valid since $\frac{1}{\sqrt{e}} < \frac{1}{\sqrt{2}}$. This remark motivates us to give a correct form of theorem 1 in [1], to verify if (3) is a simple consequence of Lagrange Theorem and if the supplementary condition ($t > (1-x)/2$) as required in (3) is necessary.

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The paper is organized as follows. In Section 2 the main results Theorem 1, Corollary 1 and Corollary 2 are presented. Corollary 1 is the correct formulation of theorem 1 in [1]. In section 3, we examine whether (3) is a simple consequence of (2) and the estimate given in Corollary 1. In section 4, Corollary 2 is used to show that the right hand-side inequality in (3) is valid for $x > 0$ and $t > 0$, improving the condition ($t > (1-x)/2$) of Theorem 1 in [2].

2. MAIN RESULTS

In this section, we give Theorem 1, Corollary 1 and Corollary 2. The proof of Theorem 1 is based on lemma 1 which is a consequence of Lagrange Theorem.

Theorem 1. Let $\alpha > 0$. Then we have:

i) If $t > 0$ and $x > \alpha$, then

$$\frac{x}{\alpha}\left(e^{\alpha}-\left(1+\frac{x}{t}\right)^{\frac{\alpha t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-\alpha} < e^x-\left(1+\frac{x}{t}\right)^t < \frac{x}{\alpha}\left(e^{\alpha}-\left(1+\frac{x}{t}\right)^{\frac{\alpha t}{x}}\right)e^{x-\alpha}. \tag{5}$$

ii) If $t > 0$ and $0 < x < \alpha$, then

$$\frac{x}{\alpha}\left(e^{\alpha}-\left(1+\frac{x}{t}\right)^{\frac{\alpha t}{x}}\right)e^{x-\alpha} < e^x-\left(1+\frac{x}{t}\right)^t < \frac{x}{\alpha}\left(e^{\alpha}-\left(1+\frac{x}{t}\right)^{\frac{\alpha t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-\alpha}. \tag{6}$$

Corollary 1.

i) If $t > 0$ and $x > 1$, then

$$x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1} < e^x-\left(1+\frac{x}{t}\right)^t < x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)e^{x-1}. \tag{7}$$

ii) If $t > 0$ and $0 < x < 1$, then

$$x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)e^{x-1} < e^x-\left(1+\frac{x}{t}\right)^t < x\left(e-\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1}. \tag{8}$$

Corollary 2. If $t > 0$ and $x > 0$, then

$$x\left(1+\frac{x}{t}\right)^t\left(1-\frac{t}{x}\ln\left(1+\frac{x}{t}\right)\right) < e^x-\left(1+\frac{x}{t}\right)^t < xe^x\left(1-\frac{t}{x}\ln\left(1+\frac{x}{t}\right)\right). \tag{9}$$

The proof of Theorem 1 is based on the following Lemma:

Lemma 1. Let $a > b > 0$ and $v > 0$. Then

i) For $v > 1$, we have

$$v(a-b)b^{v-1} < a^v-b^v < v(a-b)a^{v-1}. \tag{10}$$

ii) For $v < 1$, we have

$$v(a-b)a^{v-1} < a^v-b^v < v(a-b)b^{v-1}. \tag{11}$$

Proof of Lemma 1. Applying Lagrange Theorem to the function $f(u) = u^v$ with $b < u < a$, yields

$$f(a)-f(b) = f'(c)(a-b), \qquad (b < c < a). \tag{12}$$

As $f'(u) = vu^{v-1}$ and $f''(u) = v(v-1)u^{v-2}$, the derivative $f'(u)$ increases for $v > 1$ and decreases for $0 < v < 1$. So, for $v > 1$, (10) is a consequence of (12) and the estimate $f'(b) < f'(c) < f'(a)$. Similarly, for $0 < v < 1$, (11) is deduced from (12) and the estimate $f'(a) < f'(c) < f'(b)$.

Proof of Theorem 1. Put $v = x/\alpha$, $a = e^\alpha$ and $b = (1 + x/t)^{\alpha t/x}$ in (10) and (11) to obtain (5) and (6) respectively.

Proof of Corollary 1. Put $\alpha = 1$ in Theorem 1.

Proof of Corollary 2. To obtain (9), we let α tends to zero in (5) and we use L'Hôpital's rule to get the limit

$$\lim_{\alpha\to 0} \frac{e^\alpha - \left(1+\frac{x}{t}\right)^{\frac{\alpha t}{x}}}{\alpha} = 1 - \frac{t}{x}\ln\left(1+\frac{x}{t}\right).$$

3. COMPARAISON OF (7) AND (8) WITH (3)

Multiplying (2) by $x\, e^{x-1}$ we obtain

$$\frac{x^2e^x}{2t+2x} = x\frac{ex}{2t+2x}e^{x-1} < x\left(e - \left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)e^{x-1} < x\frac{ex}{2t+x}e^{x-1} = \frac{x^2e^x}{2t+x} \quad (x,t>0). \quad (13)$$

So, from (13) we conclude that the upper bound in (7) is better than the upper bound in (3) and the lower bound in (8) is better than the lower bound in (3). Remark that, when $0 < x < 1$, the condition $t > (1-x)/2$ in lower bound (3) can be relaxed to $t > 0$. But this is not new, since the condition $t > (1-x)/2$ concerns only the upper bound in (3). See the proof of (3) in [2].

Note that, the proof of the inferiority of upper bound in (7) over upper bound in (3) was given in [1]. Also, as (1) is not valid for $0 < x < 1$, the rigorous proof given in [1] for the superiority of lower bound in (1) over lower bound in (3) is not valid.

To compare the upper bound in (8) with the upper bound in (3) we study the sign of the function $F_x(t)$ defined by

$$F_x(t) := x\left(e - \left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1} - \frac{x^2e^x}{2t+x} \qquad (0 < x < 1). \quad (14)$$

Similarly, to compare the lower bound in (7) with the lower bound in (3) we study the sign of the function $G_x(t)$ defined by

$$G_x(t) := x\left(e - \left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-1} - \frac{x^2e^x}{2t+x+x^2} \qquad (x > 1). \quad (15)$$

The signs of $F_x(t)$ and $G_x(t)$ are studied by using the following Lemma.

Lemma 2. Let $f(t)$ be a continuous real function for $t > t_f$. Where t_f is the infimum of the real numbers s such that: $f(s+\varepsilon)$ is a nonzero real number for every $\varepsilon > 0$. In the case where $f(t)$ is a nonvanishing well defined real function for every t real, we take $t_f = -\infty$.

Suppose that

$$\lim_{t\to\infty} f(t) = l \neq 0. \tag{16}$$

Then,

i) If $l > 0$ we have

$$f(t) > 0 \qquad \text{for } t > t_f. \tag{17}$$

ii) If $l < 0$ we have

$$f(t) < 0 \quad \text{for } t > t_f. \tag{18}$$

Proof of Lemma 2. Assume that $l \neq \mp\infty$. Then from (16) we have: for every $\varepsilon > 0$ there exists a real $\tilde{t}_\varepsilon$ such that for all real t,

$$t > \tilde{t}_\varepsilon \Longrightarrow l - \varepsilon < f(t) < l + \varepsilon.$$

If $l > 0$ we take $\varepsilon < l$. So, $f(t) > 0$ for $t > \tilde{t}_\varepsilon$. The case $t_f \geq \tilde{t}_\varepsilon$ is obvious. If $t_f < \tilde{t}_\varepsilon$, then $f(t)$ remains positive whenever $t_f < t < \tilde{t}_\varepsilon$ since f is continuous for $t > t_f$. The cases $l < 0$ and $l = \mp\infty$ can be treated in the same manner.

Now, using computer software such as Maple, we obtain for $t \to \infty$ the expansions

$$t^2\left(1+\frac{x}{t}\right)^{\frac{t}{x}} F_x(t) = -\frac{1}{24}(6x-1)x^3e^{x+1} + \frac{1}{48t}(3x^2+19x-2)x^4e^{x+1} + O\left(\frac{1}{t^2}\right) \tag{19}$$

and

$$t^2\left(1+\frac{x}{t}\right)^{\frac{t}{x}} G_x(t) = \frac{1}{24}x^3e^{x+1} + O\left(\frac{1}{t}\right). \tag{20}$$

Equation (19) yields

$$\lim_{t\to\infty} t^2F_x(t) = -\frac{1}{24}(6x-1)x^3e^x. \tag{21}$$

So, for $1/6 < x < 1$, we have $\lim_{t\to\infty} t^2F_x(t) < 0$. Remark that $t_{F_x} \geq 0$ and $F_x(t)$ is continuous for $t > 0$. Thus, by Lemma 2 we obtain

$$F_x(t) < 0 \qquad \text{for } t > t_{F_x}. \tag{22}$$

Inequality (22) means that for $t > t_{F_x}$ the upper bound in (8) is better than the upper bound in (3).

Observe that for $1/6 < x < \ln(e-1)$, the function F_x admits at least one positive real root, since

$$\lim_{t\to 0^+} F_x(t) = x(e-1-e^x) > 0. \tag{23}$$

So, when $1/6 < x < \ln(e-1)$, t_{F_x} is the largest positive real root of F_x. Numerical computations show that t_{F_x} is decreasing as a function of x. For instance, $t_{F_{0.1666667}} \approx 5.2\ 10^5$, $t_{F_{0.1667}} \approx 500.2$, $t_{F_{0.167}} \approx 52.13$, $t_{F_{0.17}} \approx 5.25$, $t_{F_{0.3}} \approx 0.13$, $t_{F_{0.5}} \approx 0.0076$ and $t_{F_{\ln(e-1)}} = 0$. It seems also that $t_{F_x} = 0$ for $\ln(e-1) < x < 1$.

For $0 < x < 1/6$ the limit in (21) is positive and for $x = 1/6$ we obtain from (19) that

$$\lim_{t\to\infty} t^3 F_{1/6}(t) = \frac{5e^{1/6}}{248832}. \tag{24}$$

By Lemma 2, we conclude that for $0 < x \le 1/6$, $F_x(t) > 0$ whenever $t > t_{F_x}$. So, in this case, for $t > t_{F_x}$ the upper bound in (3) is better than the upper bound in (8). Here we observe, by using numerical computations, that $t_{F_x} = 0$.

Finally, for $x > 1$, equation (20) yields

$$\lim_{t\to\infty} t^2 G_x(t) = \frac{1}{24} x^3 e^x > 0 \tag{25}$$

and from (15) we obtain the limit

$$\lim_{t\to 0^+} G_x(t)/x = e - 1 - \frac{e^x}{1+x} =: g(x).$$

We have $g(1) = e/2 - 1 > 0$ and $g(\infty) = -\infty < 0$ with

$$g'(x) = -xe^x/(1+x)^2 < 0.$$

So, $g(x)$ has a unique real root $\tilde{x} \approx 1.428$. Then, for $x > \tilde{x}$ the function $g(x)$ is negative and according to (25), $G_x(t)$ admits at least one real root where t_{G_x} is the greater one. As $G_x(t)$ is continuous for $t > t_{G_x}$, we obtain by Lemma 2 that $G_x(t) > 0$ whenever $t > t_{G_x}$. This means that for $t > t_{G_x}$, the lower bound in (7) is better than the lower bound in (3). On the other hand in a certain interval with $t < t_{G_x}$, the lower bound in (3) is better than the lower bound in (7). Notice that the comparison with the lower bound in (5) leads to analogous results, since we have

$$\frac{x}{\alpha}\left(e^\alpha - \left(1+\frac{x}{t}\right)^{\frac{\alpha t}{x}}\right)\left(\left(1+\frac{x}{t}\right)^{\frac{t}{x}}\right)^{x-\alpha} - \frac{x^2 e^x}{2t+x+x^2} = \frac{(3\alpha-2)x^3 e^x}{24t^2} + O\left(\frac{1}{t^3}\right).$$

4. IMPROVING THE CONDITION OF THEOREM 1 IN [2]

Theorem 1 in [2] states that:

i) If $x > 0$, $t > 0$ and $t > (1-x)/2$, then

$$\frac{x^2 e^x}{2t+x+\max(x,x^2)} < e^x - \left(1+\frac{x}{t}\right)^t < \frac{x^2 e^x}{2t+x}. \tag{26}$$

ii) If $x > 0$, $t > 0$ and $t > (x-1)/2$, then

$$\frac{x^2e^{-x}}{2t-x+x^2} < e^{-x} - \left(1 - \frac{x}{t}\right)^t < \frac{x^2e^{-x}}{2t-2x+\min\ (x,x^2)}. \tag{27}$$

Remark 1. According to the proof given in [2] for the two sided estimate (26), the left (resp. the right) hand-side inequality in (26) is valid for $x > 0$ and $t > 0$ (resp. $x > 0$, $t > 0$ and $t > (1-x)/2$.

Remark 2. Observe that for $t > 0$, the domain of the function $(1 - x/t)^t$ is $t > x$. So, the conditions of (27) must be replaced by $t > x > 0$, since $x > (x-1)/2$.

Now, we show that the upper bound in (9) is better than the upper bound in (26) for $t, x > 0$. We write the difference between the two upper bounds as follows

$$\frac{x^2e^x}{2t+x} - xe^x\left(1 - \frac{t}{x}\ln\left(1+\frac{x}{t}\right)\right) = te^x\left(\frac{(x/t)^2}{2+x/t} - \frac{x}{t} + \ln\left(1+\frac{x}{t}\right)\right) = te^x h(x/t).$$

A computation of the derivative of the function h yields

$$h'(z) = \frac{z^2}{(2+z)^2(1+z)} > 0.$$

Then the function h is increasing and thus $h(z) > h(0) = 0$ for $z > 0$. Consequently, the upper bound in (9) is better than the upper bound in (26). So, this confirms that the right hand-side inequality in (26) is valid for $x > 0$ and $t > 0$. From this result and taking into account remark 1 and remark 2, we can state the following Theorem which improves the conditions of Theorem 1 in [2].

Theorem 2. If $x \neq 0$ and $t > \max\ (0, -x)$, then

$$\frac{x^2e^x}{2t+x+\max\ (x,x^2)} < e^x - \left(1+\frac{x}{t}\right)^t < \frac{x^2e^x}{2t+2x+\min\ (-x,x^2)}.$$

5. CONCLUSION

The upper and lower bounds in (3), for $x > 1$ and $0 < x < 1$ respectively, are consequences of Lagrange Theorem when derived from the corresponding upper and lower bounds in Corollary 1. The upper bound in (3), for $x, t > 0$, is also a consequence of Lagrange Theorem when derived from the upper bound in Corollary 2. Concerning the lower bound in (3) for $x > 1$, and in view of section 3, we need another adequate lower bound obtained via Lagrange Theorem and better than the lower bound in (3).

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