ORIGINAL PAPER

THE NATURAL LIFT CURVE OF THE SPHERICAL INDICATRIX OF A SPACELIKE CURVE IN MINKOWSKI 4-SPACE

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Abstract. *In this study, we determine criteria of being integral curve for the geodesic spray of the natural lift curves of the spherical indicatrices of a given spacelike curve on the tangent bundle and in Minkowski 4-space.*

Keywords: *lift curve, spherical indicatrix, Minkowski 4-space.*

1. INTRODUCTION

The concepts of the natural lift and the geodesic sprays have been originally conceived by Thorpe in 1979. He proved that the natural lift curve $\overline{\alpha}$ is an integral curve of the geodesic spray X iff the original curve is an geodesic on hypersurface M in Euclidean space [6]. Çalışkan et al. studied the natural lift curves of the spherical indicatries of a given curve and gave some important results about the original curve, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2\right)$ in Euclidean space [8]. $\overline{M}$ - integral curve of Z and $\overline{M}$ - geodesic spray are defined by Sivridağ and Çalışkan. They gave the main theorem: The natural lift $\overline{\alpha}$ of the curve (in $\overline{M}$) is an $\overline{M}$ - integral curve of the geodesic spray Z iff α is an $\overline{M}$ - geodesic in [1]. Since that time, a good deal of research has been done on the natural lift and the geodesic sprays. For example, Bilici et al. studied the natural lift curves and the geodesic sprays for the spherical indicatrices of the involute curve α^*. They gave some interesting results about the evolute curve α, depending on the assumption that the natural lift curve of the spherical indicatrices of the involute curve should be the integral curve on the tangent bundle $T\left(S^2\right)$ in [10]. Then Bilici adapted this problem for the spherical indicatrices of the involutes of a timelike curve in Minkowski 3-space [11]. Çalışkan and Ergün defined $\overline{M}$ - vector field Z, $\overline{M}$ - geodesic spray, $\overline{M}$ - integral curve of Z, $\overline{M}$ - geodesic in Minkowski 3-space [13]: The anologue of the theorem of Thorpe was given in Minkowski 3-space and in Minkowski 4- space by Ergün and Çalışkan in [3, 4]. In the literature, there have been numerous papers presented over the years on the natural lift and the geodesic sprays by using anologue of the theorem of Thorpe [5, 8, 12, 16].

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In this study, we determine some criterias of being integral curve for the geodesic spray of the natural lift curves of the spherical indicatrices of a given spacelike curve on the tangent bundle $T\left(S_1^3\right)$ and $T\left(H_0^3\right)$ in Minkowski 4-space.

2. PRELIMNARIES

To meet the requirements in the next sections, here the basic elements of the theory of curves in the space E_1^4 are briefly presented. (A more complete elementary treatment can be found in [7].)

Let Minkowski 4-space $\mathbb{R}_1^4$ be the vector space $\mathbb{R}^4$ equipped with the Lorentzian inner product g given by

$$g\left(X,X\right) = -x_1^2 + x_2^2 + x_3^2 + x_4^2$$

where $X = \left(x_1, x_2, x_3, x_4\right) \in \mathbb{R}^4$. A non-zero vector $X = \left(x_1, x_2, x_3, x_4\right) \in \mathbb{R}^4$ is said to be timelike if $g\left(X,X\right) < 0$, spacelike if $g\left(X,X\right) > 0$ and lightlike (or null) if $g\left(X,X\right) = 0$. The norm of a vector X is defined by

$$\left\|X\right\|_{IL} = \sqrt{\left|g\left(X,X\right)\right|}, \text{ [2].}$$

We denote by $\left\{T\left(t\right), N\left(t\right), B_1\left(t\right), B_2\left(t\right)\right\}$ the moving Frenet frame along the curve α. the moving Frenet frame along the curve α.

Let α be a unit speed spacelike space curve. The functions k_1, k_2 and k_3 are called the first, second and third curvature of α. Let Frenet vector fields of α be $\left\{T, N, B_1, B_2\right\}$. N is spacelike vector …eld. In this trihedron, we assume that T and B1 are spacelike vector fields and B2 is a timelike vector field. Then, Frenet formulas are given by

$$\dot{T} = k_1 N,\ \dot{N} = -k_1 T + k_2 B_1,\ \dot{B}_1 = -k_2 N + k_3 B_2,\ \dot{B}_2 = k_3 B_1, \text{ [7].}$$

Let α be a unit speed spacelike space curve. N is spacelike vector eld. In this trihedron, we assume that T and B_2 are spacelike vector fields and B_1 is a timelike vector field. Then, Frenet formulas are given by

$$\dot{T} = k_1 N,\ \dot{N} = -k_1 T + k_2 B_1,\ \dot{B}_1 = k_2 N + k_3 B_2,\ \dot{B}_2 = -k_3 B_1, \text{ [7].}$$

Let α be a unit speed spacelike space curve. N is spacelike vector field. In this trihedron, we assume that T is spacelike vector field. B_1 and B_2 are null vector fields. Then, Frenet formulas are given by

$$\dot{T} = k_1 N,\ \dot{N} = -k_1 T + k_2 B_1,\ \dot{B}_1 = k_3 B_1,\ \dot{B}_2 = -k_2 N - k_3 B_2, \text{ [7].}$$

Let α be a unit speed spacelike space curve. N is timelike vector field. In this trihedron, we assume that T , B_1 and B_2 are spacelike vector fields. Then, Frenet formulas are given by

$$\dot{T} = k_1 N\ ,\ \dot{N} = k_1 T + k_2 B_1\ ,\ \dot{B}_1 = k_2 N + k_3 B_2\ ,\ \dot{B}_2 = -k_3 B_1\ ,\ [7].$$

Let α be a unit speed spacelike space curve. N is null vector field. In thistrihedron, we assume that T and B_1 are spacelike vector fields. B_2 is null vector field. Then, Frenet formulas are given by

$$\dot{T} = k_1 N\ ,\ \dot{N} = k_2 B_1\ ,\ \dot{B}_1 = k_3 N - k_2 B_2\ ,\ \dot{B}_2 = -k_1 T - k_3 B_1\ ,\ [7].$$

Definition 1. Let M be a hypersurface in $\mathbb{R}^4_1$ and let α : $I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{dt}\left(\alpha(t)\right) = X\left(\alpha(t)\right) \text{ (for all } t \in I\text{)}$$

where X is a smooth tangent vector field on M, [2]. Thus we can write

$$TM = \bigcup_{P \in M} T_P M = \chi\left(M\right),$$

where $T_P M$ is the tangent space of M at P and $\chi\left(M\right)$ is the space of vector fields of M.

Definition 2. For any parametrized curve α : $I \to M$, $\overline{\alpha} : I \to TM$ given by

$$\overline{\alpha}(t) = \left(\alpha(t), \dot{\alpha}(t)\right) = \dot{\alpha}(t)\Big|_{\alpha(t)}$$

is called the natural lift of α on TM, [4]. Thus, we can write

$$\frac{d\overline{\alpha}}{dt} = \frac{d}{dt}\left(\dot{\alpha}(t)\Big|_{\alpha(t)}\right) = D_{\dot{\alpha}(t)}\dot{\alpha}(t)$$

where D is the Levi-Civita connection on $\mathbb{R}^4_1$.

Definition 3. A X∈ $\chi\left(TM\right)$ is called a geodesic spray if for V∈TM

$$X\left(V\right) = \varepsilon g\left(S\left(V\right), V\right) N$$

where $\varepsilon = g\left(N, N\right)$, [4].

Theorem 1. The natural lift $\overline{\alpha}$ of the curve is an integral curve α of geodesic spray X if and only if α is a geodesic on M, [4].

Theorem 2. Let be a spacelike unit speed curve in E_1^4 with curvature k_1 > 0: Then α has $k_2 \equiv$ 0 if α lies fully in a 2-dimensional subspace of E_1^4, [14].

Theorem 3. Let α be a spacelike unit speed curve in E_1^4 with a spacelike principal normal N, a spacelike binormal B_1 and with cutvatures k_1 > 0; $k_2 \neq 0$. Then has $k_3 \equiv 0$ if and only if α lies fully in spacelike hyperplane of E_1^4 , [14].

Theorem 4. Let α be a spacelike unit speed curve in E_1^4 with a spacelike (timelike) principal normal N, a timelike (spacelike) binormal B_1 and with cutvatures k_1 > 0, $k_2 \neq$ 0. Then α has $k_3 \equiv 0$ if and only if α lies fully in timelike hyperplane of E_1^4, [14].

Remark 1. Recall that a spacelike unit speed curve with spacelike principal normal N and a null binormal B_1 is called a partially null spacelike curve, [14].

Theorem 5. A partially null spacelike unit speed curve α in E_1^4 with cutvatures k_1 > 0, $k_2 \neq$ 0 lies fully in a lightlike hyperplane of E_1^4 and has $k_3 \equiv 0$, [14].

Remark 2. Recall that a spacelike unit speed curve with null principal normal is called a pseudo null spacelike curve, [14].

Theorem 6. A pseudo null spacelike unit speed curve α in E_1^4 with cutvatures k_1 > 0, $k_2 \neq$ 0 lies fully in the space E_1^4, [14].

Theorem 7. Let α be a spacelike unit speed curve in E_1^4 with a spacelike principal normal N, a spacelike binormal B_1. Then has:

(i) $k_1 = c_1$, $k_2 = c_2$, $k_3 = 0$, $c_1, c_2 \in \mathbb{R}$ if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\lambda^2}\left(0, c_2\lambda s, c_1 \sin(\lambda s), c_1 \cos(\lambda s)\right), \lambda^2 = c_1^2 + c_2^2; \text{ [14].}$$

Theorem 8. Let α be a spacelike unit speed curve in E_1^4 with a spacelike principal normal N, a timelike binormal B_1 has $k_1 = c_1$, $k_2 = c_2$, $k_3 = c_3$, c_1, c_2, $c_3 \in \mathbb{R}$ if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\lambda_1}\left(V_1 \sinh(\lambda_1 s) + V_2 \cosh(\lambda_{1s})\right) + \frac{1}{\lambda_2}\left(V_3 \sin(\lambda_2 s) - V_4 \cos(\lambda_2 s)\right),$$

with $\lambda_1^2 = \dfrac{-K + \sqrt{K^2 + 4c_1^2 c_3^2}}{2}$, $\lambda_2^2 = \dfrac{K + \sqrt{K^2 + 4c_1^2 c_3^2}}{2}$ $K = c_1^2 - c_2^2 - c_3^2$, wher V_1, V_2, V_3, V_4

are mutually orthogonal vector satisfying the aquations g (V$_1$, V$_1$) = - g (V$_2$, V$_2$) = $\frac{\lambda_2^2 - c_1^2}{\lambda_1^2 + \lambda_2^2}$, g (V$_3$, V$_3$) = g (V$_4$, V$_4$) = $\frac{\lambda_2^2 + c_1^2}{\lambda_1^2 + \lambda_2^2}$, [14].

Theorem 9. A spacelike unit speed curve α in E_1^4 with a timelike principal normal N has $k_1 = c_1$, $k_2 = c_2$, $k_3 = c_3$, $c_1, c_2, c_3 \in \mathbb{R}$ if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\lambda_1}\left(V_1 \sinh(\lambda_1 s) + V_2 \cosh(\lambda_1 s)\right) + \frac{1}{\lambda_2}\left(V_3 \sin(\lambda_2 s) - V_4 \cos(\lambda_2 s)\right),$$

with $\lambda_1^2 = \frac{-K + \sqrt{K^2 + 4c_1^2 c_3^2}}{2}$, $\lambda_2^2 = \frac{K + \sqrt{K^2 + 4c_1^2 c_3^2}}{2}$ $K = c_3^2 - c_1^2 - c_2^2$, where V$_1$, V$_2$, V$_3$, V$_4$ are mutually orthogonal vector satisfying the aquations g (V$_1$, V$_1$) = - g (V$_2$, V$_2$) = $\frac{\lambda_2^2 + c_1^2}{\lambda_1^2 + \lambda_2^2}$, g (V$_3$, V$_3$) = g (V$_4$, V$_4$) = $\frac{\lambda_2^2 - c_1^2}{\lambda_1^2 + \lambda_2^2}$, [14].

Theorem 10. A partially null spacelike unit speed curve α in E_1^4 with cutvatures $k_1 = c_1$, k_2 = constant $\neq$ 0 if and only if α is a part of a partially null spacelike helix;

$$\alpha(s) = \left(as, as, \frac{1}{c_1}\sin(c_1 s), \frac{1}{c_1}\cos(c_1 s)\right), a \in \mathbb{R}, [14].$$

Theorem 11. A pseudo null spacelike unit speed curve α in E_1^4. Then α has

(i) $k_1 = 1$, $k_2 = c_2$, $k_3 = 0$, $c_2 \in \mathbb{R}$ if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\sqrt{2c_2}}\left(\cosh\left(\sqrt{c_2 s}\right), \sinh\left(\sqrt{c_2 s}\right), \sin\left(\sqrt{c_2 s}\right), \cos\left(\sqrt{c_2 s}\right)\right);$$

(ii) $k_1 = 1$, $k_2 = c_2$, $k_3 = c_3$, $c_2, c_3 \in \mathbb{R}$ if and only if α can be parameterized by

$$\alpha(s) = \frac{1}{\lambda_1}\left(V_1 \sinh(\lambda_1 s) + V_2 \cosh(\lambda_1 s)\right) + \frac{1}{\lambda_2}\left(V_3 \sin(\lambda_2 s) - V_4 \cos(\lambda_2 s)\right),$$

with $\lambda_1^2 = K + \sqrt{K^2 + c_2^2}$, $\lambda_2^2 = -K + \sqrt{K^2 + c_2^2}$ $K = c_2 c_3$, where V$_1$, V$_2$, V$_3$, V$_4$ are mutually orthogonal vector satisfying the aquations g (V$_1$, V$_1$) = - g (V$_2$, V$_2$) = $\frac{\lambda_2^2}{\lambda_1^2 + \lambda_2^2}$, g (V$_3$, V$_3$) = g (V$_4$, V$_4$) = $\frac{\lambda_1^2}{\lambda_1^2 + \lambda_2^2}$, [14].

3. THE NATURAL LIFT CURVE OF THE SPHERICAL INDICATRIX OF A SPACELIKE CURVE IN MINKOWSKI 4-SPACE

Let D, $\bar{D}$, $\bar{\bar{D}}$ and $\tilde{D}$ be connections in $\mathbb{R}_1^4$, S_1^3, H_0^3 and Λ_1^3 respectively and ξ be a unit normal vector field of S_1^3, H_0^3 and Λ_1^3, respectively. Then Gauss Equations are given by the followings

$$D_x Y = \bar{D}_x Y + \varepsilon g\left(S\left(X\right), Y\right)\xi,$$
$$D_x Y = \bar{\bar{D}}_x Y + \varepsilon g\left(S\left(X\right), Y\right)\xi,$$
$$D_x Y = \tilde{D}_x Y + \varepsilon g\left(S\left(X\right), Y\right)\xi,$$

where $\varepsilon = g\left(\xi, \xi\right)$ and $S\left(X\right) = -D_x \xi$ is the shape operator of S_1^3, H_0^3, Λ_1^3, respectively.

3.1. Let α be a unit speed spacelike space curve with timelike B_2.

N is spacelike vector field. In this trihedron, we assume that T and B_1 are spacelike vector fields and B_2 is a timelike vector field .

3.1.1. The natural lift of the spherical indicatrix of tangent vector of α.

Let α_T be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_T$ be the natural lift of the curve α_T. If $\bar{\alpha}_T$ is an integral curve of the geodesic spray, then from Theorem we have

$$\bar{\bar{D}}_{\dot{\alpha}_T} \dot{\alpha}_T = 0$$

that is

$$\left(k_1^2 - k_1\right)T + \frac{\dot{k}_1}{k_1}N + k_2 B_1 = 0.$$

Since T, N, B_1, B_2 are linearly independent we have

$$k_1 = 1,\ k_2 = 0.$$

Thus, we obtain the following corollary:

Corollary 1. If the natural lift $\bar{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then lies fully in a 2-dimensional subspace of $\mathbb{R}_1^4$.

3.1.2. The natural lift of the spherical indicatrix of principal normal vector of α.

Let α_N be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_N$ be the natural lift of the curve α_N. If $\bar{\alpha}_N$ is an integral curve of the geodesic spray, then because of Theorem we have

$$\bar{D}_{\dot{\alpha}_N}\dot{\alpha}_N = 0$$

that is

$$\frac{-\dot{k}_1}{\sqrt{k_1^2+k_2^2}}T - \left(\frac{k_1^2+k_2^2}{\sqrt{k_1^2+k_2^2}} + k_1^2 + k_2^2\right)N + \frac{\dot{k}_2}{\sqrt{k_1^2+k_2^2}}B_1 + \frac{k_2k_3}{\sqrt{k_1^2+k_2^2}}B_2 = 0$$

Since T, N, B_1, B_2 are linearly independent we can state the following corollaries:

Corollary 2. If the natural lift $\bar{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then lies fully in a 2-dimensional subspace or spacelike hyperplane of $\mathbb{R}_1^4$.

3.1.3. The natural lift of the spherical indicatrix of the firsth binormal vectors of α.

Let α_{B_1} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_1}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_1}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\bar{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1} = 0$$

that is

$$\frac{k_1k_2}{\sqrt{k_2^2-k_3^2}}T - \frac{\dot{k}_2}{\sqrt{k_2^2-k_3^2}}N + \left(\frac{-k_2^2+k_3^2}{\sqrt{k_2^2-k_3^2}} + k_2^2 - k_3^2\right)B_1 + \frac{\dot{k}_2}{\sqrt{k_1^2+k_2^2}}B_1 + \frac{\dot{k}_3}{\sqrt{k_2^2-k_3^2}}B_2 = 0$$

Because T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 3. If the natural lift $\bar{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then we have k_1 = 0 and k_2 = constant $\neq 0$, k_3 = constant $\neq$ 0. Therefore there is no such curve satisfying this condition.

3.1.4. The natural lift of the spherical indicatrix of the second binormalvectors of α.

Let α_{B_2} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_2}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_2}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\bar{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B2} = 0$$

that is

$$-k_2N + \frac{\dot{k}_3}{k_3}B_1 + B_2 = 0\,.$$

Because T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 4. If the natural lift $\overline{\alpha}_{B_2}$ of α_{B_2} is an integral curve of the geodesic on the tangent bundle $T\left(H_0^3\right)$ then we have k_2 = 0 and k_3 = 1. Therefore there is no such curve satisfying this condition.

***3.2. Let α be a unit speed spacelike space curve with timelike** B_1.*

N is spacelike vector field. In this trihedron, we assume that T and B_2 are spacelike vector fields and B_1 is a timelike vector field.

3.2.1. The natural lift of the spherical indicatrix of the tangent vectors of α.

Let α_T be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_T$ be the natural lift of the curve α_T . If $\overline{\alpha}_T$ is an integral curve of the geodesic spray, then from Theorem we have

$$\overline{\overline{D}}_{\dot{\alpha}_T}\dot{\alpha}_T = 0$$

that is

$$D_{\dot{\alpha}_T}\dot{\alpha}_T = \overline{\overline{D}}_{\dot{\alpha}_T}\dot{\alpha}_T + \varepsilon g\left(S\left(\dot{\alpha}_T\right),\dot{\alpha}_T\right)T \ .$$

Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 5. If the natural lift $\overline{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(H_0^3\right)$, then α lies fully in a 2-dimensional subspace of $\mathbb{R}_1^4$.

3.2.2. The natural lift of the spherical indicatrix of the principal normal vectors of α.

Let α_N be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_N$ be the natural lift of the curve α_N. If $\overline{\alpha}_N$ is an integral curve of the geodesic spray, then because of Theorem we have

$$\overline{D}_{\dot{\alpha}_N}\dot{\alpha}_N = 0$$

that is

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \overline{D}_{\dot{\alpha}_N}\dot{\alpha}_N + \varepsilon g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right)\xi$$

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \varepsilon g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right)N \ .$$

where $\varepsilon = g\left(\xi,\xi\right)$ and $\xi = N$. Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 6. If the natural lift $\overline{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then α lies fully in timelike hyperplane of $\mathbb{R}_1^4$.

3.2.3. The natural lift of the spherical indicatrix of the firsth binormal vectors of α.

Let α_{B_1} be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_{B_1}$ be the natural lift of the curve α_{B_1}. If $\overline{\alpha}_{B_1}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\overline{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=0$$

that is

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\overline{D}_{\dot{\alpha}_N}\dot{\alpha}_{B_1}+\varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)\xi$$

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)B_1.$$

where $\varepsilon=g\left(\xi,\xi\right)$ and $\xi=B_1$. Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 7. If the natural lift $\overline{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then we have k_1 = 0 and k_2 = constant, k_3 = constant. Therefore there is no such curve satisfying this condition.

3.2.4. The natural lift of the spherical indicatrix of the second binormal vectors of α.

Let α_{B_2} be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_{B_2}$ be the natural lift of the curve α_{B_1}. If α_{B_1} is an integral curve of the geodesic spray, then by using Theorem we have

$$\overline{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=0$$

that is

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\overline{D}_{\dot{\alpha}_2}\dot{\alpha}_{B_2}+\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)\xi$$

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)B_2$$

where $\varepsilon=g\left(\xi,\xi\right)$ and $\xi=B_2$. Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 8. If the natural lift $\overline{\alpha}_{B_2}$ of α_{B_2} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then we have k_2 = 0 and k_3 = 1. Therefore there is no such curve satisfying this condition.

3.3. Let α be a unit speed patially null spacelike space curve.

N is spacelike vector field. In this trihedron, we assume that T is spacelike vector field. B_1 and B_2 are null vector fields.

3.3.1. The natural lift of the spherical indicatrix of the tangent vectors of α.

Let α_T be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_T$ be the natural lift of the curve α_T . If $\bar{\alpha}_T$ is an integral curve of the geodesic spray, then from Theorem we have

$$\overline{\overline{D}}_{\dot{\alpha}_T}\dot{\alpha}_T = 0$$

that is

$$D_{\dot{\alpha}_T}\dot{\alpha}_T = \overline{\overline{D}}_{\dot{\alpha}_T}\dot{\alpha}_T + \varepsilon g\left(S\left(\dot{\alpha}_T\right), \dot{\alpha}_T\right)\xi$$

$$D_{\dot{\alpha}_T}\dot{\alpha}_T = \varepsilon g\left(S\left(\dot{\alpha}_T\right), \dot{\alpha}_T\right)T .$$

Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 9. If the natural lift $\bar{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(H_0^3\right)$, then lies fully in a 2-dimensional subspace of $\mathbb{R}_1^4$.

3.3.2. The natural lift of the spherical indicatrix of the principal normal vectors of α.

Let α_N be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_N$ be the natural lift of the curve α_N . If $\bar{\alpha}_N$ is an integral curve of the geodesic spray, then because of Theorem we have

$$\overline{D}_{\dot{\alpha}_N}\dot{\alpha}_N = 0$$

that is

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \overline{D}_{\dot{\alpha}_N}\dot{\alpha}_N + \varepsilon g\left(S\left(\dot{\alpha}_N\right), \dot{\alpha}_N\right)\xi$$

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \varepsilon g\left(S\left(\dot{\alpha}_N\right), \dot{\alpha}_N\right)N$$

Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 10. If the natural lift $\bar{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$,then α lies fully in a lightlike hyperplane of $\mathbb{R}_1^4$.

3.3.3. The natural lift of the spherical indicatrix of the firsth binormal vectors of α.

Let α_{B_1} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_1}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_1}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\tilde{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=0$$

that is

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\tilde{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}+\varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)\xi$$

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)B_1$$

Because T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 11. If the natural lift $\bar{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T\left(\Lambda_1^3\right)$, then α is a part of a partially null spacelike helix;

$$\alpha(s)=\left(as,as,\frac{1}{c_1}\sin(c_1 s),\frac{1}{c_1}\cos(c_1 s)\right),a\in\mathbb{R}.$$

Corollary 12. If the natural lift $\bar{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then α_{B_1} a is geodesic on $\mathbb{R}_1^4$.

3.3.4. The natural lift of the spherical indicatrix of the second binormal vectors of α.

Let α_{B_2} be the spherical indicatrix of α tangent vectors of and $\bar{\alpha}_{B_2}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_2}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\tilde{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=0$$

that is

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\tilde{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}+\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)\xi$$

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)B_2$$

Because T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 13. If the natural lift $\bar{\alpha}_{B_2}$ of α_{B_2} is an integral curve of the geodesic on the tangent bundle $T\left(\Lambda_1^3\right)$, then we have $k_2 = 0$ and $k_3 = 1$. Therefore there is no such curve satisfying this condition.

3.4. Let α be a unit speed spacelike space curve with timelike N.

N is timelike vector field. In this trihedron, we assume that T, B_1 and B_2 are spacelike vector fields.

3.4.1. The natural lift of the spherical indicatrix of the tangent vectors of α.

Let α_T be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_T$ be the natural lift of the curve α_T. If $\overline{\alpha}_T$ is an integral curve of the geodesic spray, then from Theorem we have

$$\overline{\overline{D}}_{\dot{\alpha}_T}\dot{\alpha}_T = 0$$

that is

$$D_{\dot{\alpha}_T}\dot{\alpha}_T = \overline{\overline{D}}_{\dot{\alpha}_T}\dot{\alpha}_T + \varepsilon g\left(S\left(\dot{\alpha}_T\right),\dot{\alpha}_T\right)\xi$$

$$D_{\dot{\alpha}_T}\dot{\alpha}_T = \varepsilon g\left(S\left(\dot{\alpha}_T\right),\dot{\alpha}_T\right)T .$$

Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 14. If the natural lift $\overline{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(H_0^3\right)$, then α lies fully in a 2-dimensional subspace of $\mathbb{R}_1^4$.

3.4.2. The natural lift of the spherical indicatrix of the principal normal vectors of α.

Let α_N be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_N$ be the natural lift of the curve α_N. If $\overline{\alpha}_N$ is an integral curve of the geodesic spray, then because of Theorem we have

$$\overline{D}_{\dot{\alpha}_N}\dot{\alpha}_N = 0$$

that is

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \overline{D}_{\dot{\alpha}_N}\dot{\alpha}_N + \varepsilon g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right)\xi$$

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \varepsilon g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right)N .$$

Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 15. If the natural lift $\overline{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then α lies fully in timelike hyperplane of $\mathbb{R}_1^4$.

3.4.3. The natural lift of the spherical indicatrix of the firsth binormal vectors of α.

Let α_{B_1} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_1}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_1}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\bar{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=0$$

that is

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\bar{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}+\varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)\xi$$

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)B_1.$$

Because T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 16. If the natural lift $\bar{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then we have $k_1 = 0$ and k_2 = constant, k_3 = constant. Therefore there is no such curve satisfying this condition.

3.4.4. The natural lift of the spherical indicatrix of the second binormal vectors of α.

Let α_{B_2} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_2}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_2}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\bar{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=0$$

that is

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\bar{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}+\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)\xi$$

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)B_2.$$

Because T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 17. If the natural lift $\bar{\alpha}_{B_2}$ of α_{B_2} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then we have $k_2 = 0$ and $k_3 = 1$. Therefore there is no such curve satisfying this condition.

3.5. Let α be a unit speed pseudo null spacelike space curve.

N is null vector field. In this trihedron, we assume that T and B_1 are spacelike vector fields. B_2 is null vector field.

3.5.1. The natural lift of the spherical indicatrix of the tangent vectors of α.

Let α_T be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_T$ be the natural lift of the curve α_T. If $\bar{\alpha}_T$ is an integral curve of the geodesic spray, then from Theorem we have

$$\overline{\overline{D}}_{\dot{\alpha}_T}\dot{\alpha}_T = 0$$

that is

$$D_{\dot{\alpha}_T}\dot{\alpha}_T = \overline{\overline{D}}_{\dot{\alpha}_T}\dot{\alpha}_T + \varepsilon g\left(S\left(\dot{\alpha}_T\right),\dot{\alpha}_T\right)\xi$$

$$D_{\dot{\alpha}_T}\dot{\alpha}_T = \varepsilon g\left(S\left(\dot{\alpha}_T\right),\dot{\alpha}_T\right)T .$$

Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 18. If the natural lift $\bar{\alpha}_T$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(H_0^3\right)$, then lies fully in a 2-dimensional subspace of $\mathbb{R}_1^4$.

3.5.2. The natural lift of the spherical indicatrix of the principal normal vectors of α.

Let α_N be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_N$ be the natural lift of the curve α_N. If $\bar{\alpha}_N$ is an integral curve of the geodesic spray, then because of Theorem we have

$$\tilde{D}_{\dot{\alpha}_N}\dot{\alpha}_N = 0$$

that is

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \tilde{D}_{\dot{\alpha}_N}\dot{\alpha}_N + \varepsilon g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right)\xi$$

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \varepsilon g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right)N .$$

Since T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 19. If the natural lift $\bar{\alpha}_N$ of α_N is an integral curve of the geodesic on the tangent bundle $T\left(\Lambda_1^3\right)$, then α lies fully in a 2-dimensional subspace of $\mathbb{R}_1^4$.

3.5.3. The natural lift of the spherical indicatrix of the firsth binormal vectors of α.

Let α_{B_1} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_1}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_1}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\bar{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1} = 0$$

that is

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1} = \bar{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1} + \varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)\xi$$

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1} = \varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)B_1$$

Because T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 20. If the natural lift $\bar{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then lies fully in a 2-dimensional subspace of $\mathbb{R}_1^4$.

Corollary 21. If the natural lift $\bar{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then α_{B_1} a is geodesic on $\mathbb{R}_1^4$.

3.5.4. The natural lift of the spherical indicatrix of the second binormal vectors of α.

Let α_{B_2} be the spherical indicatrix of tangent vectors of α and $\bar{\alpha}_{B_2}$ be the natural lift of the curve α_{B_1}. If $\bar{\alpha}_{B_2}$ is an integral curve of the geodesic spray, then by using Theorem we have

$$\tilde{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=0$$

that is

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\tilde{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}+\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)\xi$$

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)B_2.$$

Because T, N, B_1, B_2 are linearly independent, we have the following corollary:

Corollary 22. If the natural lift $\bar{\alpha}_{B_2}$ of α_{B_2} is an integral curve of the geodesic on the tangent bundle $T\left(\Lambda_1^3\right)$, then can be parameterized by

$$\alpha(s)=\frac{1}{\sqrt{2c_2}}\left(\cosh\left(\sqrt{c_2 s}\right),\sinh\left(\sqrt{c_2 s}\right),\sin\left(\sqrt{c_2 s}\right),\cos\left(\sqrt{c_2 s}\right)\right);$$

Corollary 23. If the natural lift $\bar{\alpha}_{B_2}$ of α_{B_2} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$, then α_{B_2} a is geodesic on $\mathbb{R}_1^4$.

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