ORIGINAL PAPER

SURFACE FAMILY WITH A COMMON NATURAL ASYMPTOTIC LIFT

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Manuscript received: 02.05.2015; Accepted paper: 12.06.2015;
Published online: 30.06.2015.

Abstract. *In the present paper, we find a surface family possessing the natural lift of a given curve as a common asymptotic curve. We express necessary and sufficient conditions for the given curve such that its natural lift is an asymptotic curve on any member of the surface family. We present important results for ruled surfaces. Finally, we illustrate the method with some examples.*

Keywords: *Asymptotic curve, Surface family, Natural lift curve.*

1. INTRODUCTION AND PRELIMINARIES

Wang et.al. [1] is the first to handle the problem of finding a surface family possessing a given curve as a special curve. They obtain surfaces interpolating a given curve as a common geodesic. Kasap et.al. [2] generalized the marching-scale functions of Wang and obtained a larger family of surfaces. In 2011, the common curve is changed from geodesic to line of curvature by Li et.al. [3]. Bayram et.al. [4] tackled the problem of constructing surfaces passing through a given asymptotic curve.

Inspired with the above papers, we search for a surface family possessing the natural lift of a given curve as a common asymptotic curve. We obtain the sufficient condition for the resulting surface to be a ruled surface and present a result for developable surfaces.

We start with giving some background about the subject. A parametric curve $\alpha(s),\ L_1 \le s \le L_2,$ is a curve on a surface $P(s,t)$ in IR^3 that has a constant s or t -parameter value. In this paper, α' denotes the derivative of α with respect to arc length parameter s and we assume that α is a regular curve with $\alpha''(s) \ne 0,\ L_1 \le s \le L_2$. For every point of $\alpha(s)$, the set $\{T(s), N(s), B(s)\}$ is called the Frenet frame along $\alpha(s)$, where $T(s) = \alpha'(s),\ N(s) = \frac{\alpha''}{\|\alpha''\|}$ and $B(s) = T(s) \times N(s)$ are the unit tangent, principal normal, and binormal vectors of the curve at the point $\alpha(s)$, respectively. Derivative formulas of the Frenet frame is governed by the relations

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$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{1}$$

where $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle B'(s), N(s)\rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively [5].

Let M be a surface in IR^3 and let $\alpha : I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)), \ \forall\, t \in I,$$

where X is a smooth tangent vector field on M. We have

$$TM = \bigcup_{P\in M} T_P M = \chi(M),$$

where $T_P M$ is the tangent space of M at P and $\chi(M)$ is the space of tangent vector fields on M.

For any parametrized curve $\alpha : I \to M$, $\overline{\alpha} : I \to TM$ given by

$$\overline{\alpha}(s) = \left(\alpha(s), \alpha'(s)\right) = \alpha'(s)|_{\alpha(s)} \tag{2}$$

is called the natural lift of α on TM [6] $\cdot$ Thus, we can write

$$\tfrac{d\overline{\alpha}}{ds} = \tfrac{d}{ds}\left(\alpha'(s)|_{\alpha(s)}\right).$$

If a rigid body moves along a unit speed curve $\alpha(s)$, then the motion of the body consists of translation along α and rotation about α. The rotation is determined by an angular velocity vector ω which satisfies $T' = \omega \times T$, $N' = \omega \times N$ and $B' = \omega \times B$. The vector ω is called the *Darboux vector*. In terms of Frenet vectors T, N and B, Darboux vector is given by $\omega = \tau T + \kappa B$ [7]. Also, we have $\kappa = \|\omega\|\cos\theta,\ \tau = \|\omega\|\sin\theta,$ where θ is the angle between the Darboux vector and the binormal vector $B(s)$ of α. Observe that $\theta = \arctan\frac{\tau}{\kappa}\ (Fig.1)$.

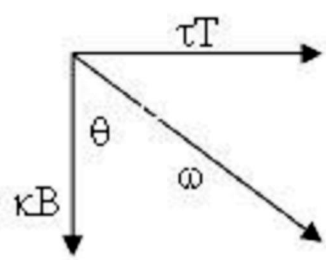

Figure 1. Darboux vector ω, tangent vector T and binormal vector B of α.

Let $\alpha(s),\ L_1 \le s \le L_2,$ be an arc length curve and $\overline{\alpha}(s),\ L_1 \le s \le L_2,$ be the natural lift of α. Then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{3}$$

where $\{T(s), N(s), B(s)\}$ and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ are Frenet frames of the curves α and $\bar{\alpha}$, respectively, and θ is the angle between the Darboux vector and the binormal vector of α .

2. SURFACE FAMILY WITH A COMMON NATURAL ASYMPTOTIC LIFT

Suppose we are given a 3-dimensional parametric curve $\alpha(s), L_1 \le s \le L_2,$ in which s is the arc length and $\|\alpha''(s)\| \ne 0, L_1 \le s \le L_2$. Let $\bar{\alpha}(s), L_1 \le s \le L_2,$ be the natural lift of the given curve $\alpha(s)$.

Surface family that interpolates $\bar{\alpha}(s)$ as a common curve is given in the parametric form as

$$P(s,t) = \bar{\alpha}(s) + u(s,t)\bar{T}(s) + v(s,t)\bar{N}(s) + w(s,t)\bar{B}(s), \tag{4}$$

where $u(s,t)$, $v(s,t)$ and $w(s,t)$ are C^1 functions, called *marching-scale functions,* and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ is the Frenet frame of the curve $\bar{\alpha}$. Using Eqn. (3) we can express Eqn. (4) in terms of Frenet frame $\{T(s), N(s), B(s)\}$ of the curve α as

$$\begin{aligned} P(s,t) = \bar{\alpha}(s) &+ \big(w(s,t)\sin\theta - v(s,t)\cos\theta\big)T(s) \\ &+ u(s,t)N(s) + \big(v(s,t)\sin\theta + w(s,t)\cos\theta\big)B(s), \end{aligned} \tag{5}$$

where $L_1 \le s \le L_2,\ T_1 \le t \le T_2.$

Remark1: *Observe that choosing different marching-scale functions yields different surfaces possessing* $\bar{\alpha}(s)$ *as a common curve.*

Our goal is to find the necessary and sufficient conditions for which the curve $\bar{\alpha}(s)$ is isoparametric and asymptotic curve on the surface $P(s,t)$. Firstly, as $\bar{\alpha}(s)$ is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0,\ L_1 \le s \le L_2,\ T_1 \le t_0 \le T_2. \tag{6}$$

Secondly the curve $\bar{\alpha}$ is an asymptotic curve on the surface $P(s,t)$ if and only if along the curve the surface normal vector field $n(s,t_0)$ is orthogonal to the principal normal vector field $\bar{N}$ of the curve $\bar{\alpha}$. Equivalently, $\bar{\alpha}$ is asymptotic if and only if $n(s,t_0)$ is parallel to the binormal vector field $\bar{B}$. The normal vector of $P(s,t)$ can be written as

$$n(s,t)=\frac{\partial P(s,t)}{\partial s}\times\frac{\partial P(s,t)}{\partial t}. \tag{7}$$

Along the curve $\bar{\alpha}$ the normal vector can be expressed as

$$n(s,t_0)=\kappa\left[-\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s)+\frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right],$$

where κ is the curvature of the curve α. Since $\kappa(s)\neq 0,\ L_1\leq s\leq L_2,$ the curve $\bar{\alpha}$ is an asymptotic curve on the surface $P(s,t)$ if and only if

$$\frac{\partial w}{\partial t}(s,t_0)=0,\ \frac{\partial v}{\partial t}(s,t_0)\neq 0.$$

So, we can present :

Theorem 2: *Let $\alpha(s),\ L_1\leq s\leq L_2,$ be a unit speed curve with nonvanishing curvature and $\bar{\alpha}(s),\ L_1\leq s\leq L_2,$ be its natural lift. $\bar{\alpha}$ is an asymptotic curve on the surface* (4) *if and only if*

$$\begin{cases} u(s,t_0)=v(s,t_0)=w(s,t_0)=\frac{\partial w}{\partial t}(s,t_0)\equiv 0, \\ \frac{\partial v}{\partial t}(s,t_0)\neq 0, \end{cases} \tag{8}$$

where $L_1\leq s\leq L_2,\ T_1\leq t,\ t_0\leq T_2$ *(* t_0 *fixed).*

Corollary3: *Let $\alpha(s),\ L_1\leq s\leq L_2,$ be a unit speed curve with nonvanishing curvature and $\bar{\alpha}(s),\ L_1\leq s\leq L_2,$ be its natural lift. If*

$$u(s,t)=v(s,t)=t-t_0,\ w(s,t)\equiv 0, \tag{9}$$

where $L_1\leq s\leq L_2,\ T_1\leq t,\ t_0\leq T_2\ (t_0 \text{ fixed})$ *then* (4) *is a ruled surface possessing* $\bar{\alpha}$ *as an asymptotic curve.*

Proof: By taking marching scale functions as $u(s,t)=v(s,t)=(t-t_0),\ w(s,t)\equiv 0,$ the surface (4) takes the form

$$P(s,t)=\bar{\alpha}(s)+(t-t_0)\left[\bar{T}(s)+\bar{N}(s)\right] \tag{10}$$

which is a ruled surface satisfies Eqn. (8).

Corollary4: *Ruled surface* (10) *is developable if and only if $\alpha(s)$ is a unit speed helix.*

Corollary5: *If the ruled surface* (10) *is developable then it is a part of a plane.*

3.EXAMPLES

Example 6: *Let* $\alpha(s)=\left(\frac{4}{5}\cos s, 1-\sin s, -\frac{3}{5}\cos s\right)$ *be a unit speed curve. Then, it is easy to show that*

$$T(s)=\left(-\frac{4}{5}\sin s,-\cos s,\frac{3}{5}\sin s\right),$$
$$N(s)=\left(-\frac{4}{5}\cos s,\sin s,\frac{3}{5}\cos s\right),$$
$$B(s)=\left(-\frac{3}{5},0,-\frac{4}{5}\right),$$
$$\kappa=1,\ \tau=0,\ \theta=0.$$

We have

$$\bar{\alpha}(s)=\left(-\frac{4}{5}\sin s,-\cos s,\frac{3}{5}\sin s\right)$$

as the natural lift of α *with Frenet vectors*

$$\bar{T}(s)=\left(-\frac{4}{5}\cos s,\sin s,\frac{3}{5}\cos s\right),$$
$$\bar{N}(s)=\left(\frac{4}{5}\sin s,\cos s,-\frac{3}{5}\sin s\right),$$
$$\bar{B}(s)=\left(-\frac{3}{5},0,-\frac{4}{5}\right).$$

If we choose $u(s,t)=v(s,t)=t,\ w(s,t)\equiv 0$ *and* $t_0=0,$ *then Eqn.* (9) *is satisfied and we get the ruled surface*

$$P_1(s,t)=\bar{\alpha}(s)+t\left[\bar{T}(s)+\bar{N}(s)\right]$$
$$=\left(-\frac{4}{5}(\sin s+t\cos s-t\sin s),t(\sin s+\cos s)-\cos s,\right.$$
$$\left.\frac{3}{5}(\sin s+t\cos s-t\sin s)\right),$$

$-0\leq s\leq 2,\ -0,5\leq t\leq 0,5,$ *possessing* $\bar{\alpha}$ *as an asymptotic curve* $(Fig.\,2)$.

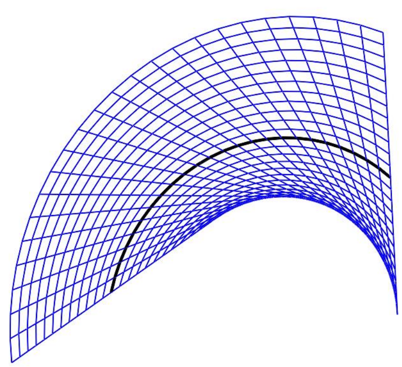

Figure 2. Ruled surface $P_1(s,t)$ as a member of the surface family and its common natural asymptotic lift $\bar{\alpha}$

For the same curve, if we take $u(s,t)=e^{2t}-1,\ v(s,t)=t,\ w(s,t)\equiv 0$ and $t_0=0$ then Eqn. (8) is satisfied and we obtain the surface

$$P_2(s,t)=\bar{\alpha}(s)+\left(e^{2t}-1\right)\bar{T}(s)+t\bar{N}(s)$$
$$=\Bigg(-\frac{4}{5}\left(\left(e^{2t}-1\right)\cos s+(1-t)\sin s\right),\left(e^{2t}-1\right)\sin s+(t-1)\cos s,$$
$$\frac{3}{5}\left(\left(e^{2t}-1\right)\cos s+(1-t)\sin s\right)\Bigg),$$

$0\le s\le 3,\ -1\le t\le 0$ interpolating $\bar{\alpha}$ as the natural asymptotic lift $(Fig.\,3)$.

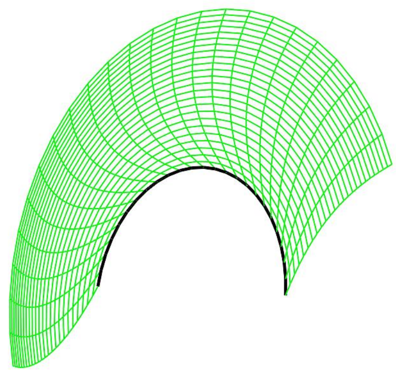

Figure 3. $P_2(s,t)$ **as a member of the surface family and its common natural asymptotic lift** $\bar{\alpha}$

Example 7: *Let* $\alpha(s)=\left(\frac{\sqrt{3}}{2}\sin s,\frac{s}{2},\frac{\sqrt{3}}{2}\cos s\right)$ *be an arc length helix. One can show that*

$$T(s)=\left(\frac{\sqrt{3}}{2}\cos s,\frac{1}{2},-\frac{\sqrt{3}}{2}\sin s\right),$$
$$N(s)=(-\sin s,0,-\cos s),$$
$$B(s)=\left(-\frac{1}{2}\cos s,\frac{\sqrt{3}}{2},\frac{1}{2}\sin s\right),$$
$$\kappa=\frac{\sqrt{3}}{2},\ \tau=\frac{1}{2},\ \theta=\frac{\pi}{6}.$$

We obtain

$$\bar{\alpha}(s)=\left(\frac{\sqrt{3}}{2}\cos s,\frac{1}{2},-\frac{\sqrt{3}}{2}\sin s\right)$$

as the natural lift of α with Frenet vectors

$$\bar{T}(s)=(-\sin s,0,-\cos s),$$
$$\bar{N}(s)=(-\cos s,0,\sin s),$$
$$\bar{B}(s)=(0,1,0).$$

Choosing marching scale functions as $u(s,t)=s^2t,\ v(s,t)=\sin t,\ w(s,t)=t^2$ and $t_0=0,$ we get the surface

$$P_3(s,t)=\bar{\alpha}(s)+s^2t\bar{T}(s)+\sin t\bar{N}(s)+t^2\bar{B}(s)$$
$$=\left(\frac{\sqrt{3}}{2}\cos s-s^2t\sin s-(\cos s)\sin t,\frac{1}{2}+t^2,\right.$$
$$\left.-\frac{\sqrt{3}}{2}\sin s-s^2t\cos s+(\sin s)\sin t\right),$$

$-2,5\le s\le 2,5,-0,5\le t\le 0,$ satisfying Eqn. (8) possessing $\bar{\alpha}$ as a common natural astymptotic lift $(Fig.4)$.

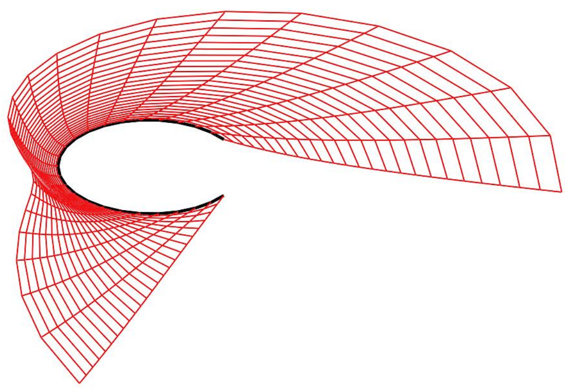

Figure 4. $P_3(s,t)$ **as a member of the surface family and its common natural asymptotic lift** $\bar{\alpha}$

If we let $u(s,t)=t\sin s,\ v(s,t)=t\cos s,\ w(s,t)=t^2,$ then Eqn. (8) is satisfied and we have

$$P_4(s,t)=\bar{\alpha}(s)+t\sin s\bar{T}(s)+t\cos s\bar{N}(s)+t^2\bar{B}(s)$$
$$=\left(\frac{\sqrt{3}}{2}\cos s-t,\frac{1}{2}+t^2,-\frac{\sqrt{3}}{2}\sin s\right),$$

$-2,5\le s\le 2,5,\ -1\le t\le 0,$ as a member of the surface family possessing $\bar{\alpha}$ as a common natural asymptotic lift $(Fig.5)$.

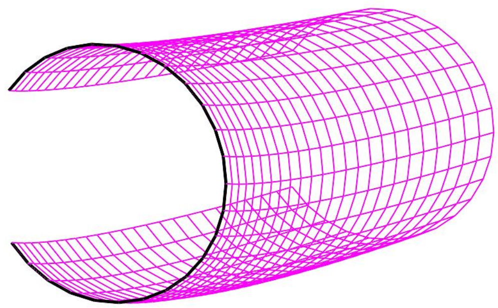

Figure 5. $P_5(s,t)$ **as a member of the surface family and its common natural asymptotic lift** $\bar{\alpha}$

4. Acknowledgments

The first author would like to thank TUBITAK (The Scientific and Technological Research Council of Turkey) for their financial supports during his doctorate studies.

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