ORIGINAL PAPER

THE NATURAL LIFT CURVE OF THE SPHERICAL INDICATRIX OF A TIMELIKE CURVE IN MINKOWSKI 4-SPACE

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Manuscript received: 16.04.2015; Accepted paper: 22.05.2015;
Published online: 30.06.2015.

Abstract. *In this study,some interesting results about the original timelike curve were obtained depending on the assumption that the natural lift curves should be the integral curve of the geodesic spray on the tangent bundle* $T\left(S_1^3\right)$ *and* $T\left(H_0^3\right)$.

Keywords: *Natural Lift, Geodesic Sprays.*

1. INTRODUCTION AND PRELIMINARIES

In differential geometry, there are many important consequences and properties of curve [2, 3, 14] Thorpe gave the concepts of the natural lift curve and geodesic spray in [9]. Thorpe provied the natural lift $\overline{\alpha}$ of the curve α is an integralcurve of the geodesic spray iff α is an geodesic on M in [9].Çalėkkan, Sivridaĝ and Hacėsalihoĝlu studied the natural lift curves of the spherical indicatries of tangent, principal normal, binormal vectors and fixed centrode of a curve in [12].They gave some interesting results about the original curve were obtaied, depending on the assumption that the natural lift curve should be the integral curve of the geodesic spray on the tangent bundle $T\left(S^2\right)$ in [12]. $\overline{M}$ -integral curve of Z and $\overline{M}$ -geodesic spray are defined by Sivridaĝ and Çalėkkan. They gave the main theorem: The natural lift $\overline{\alpha}$ of the curve α (in $\overline{M}$) is an $\overline{M}$ -integral curve of the geodesic spray Z iff α is an $\overline{M}$ -geodesic in [1]. Bilici, Çalėkkan and Aydemir studied $\left(\alpha,\alpha^*\right)$ being the pair of evolute-involute curves,the natural lift curve of the spherical indicatries of tangent, principal normal, binormal vectors of the involute curve α^* They gave some interesting results about the evolute curve α were obtained, depending on the assumption that the natural lift curve of the spherical indicatrices of the involute α^* should be the integral curve on the tangent bundle $T\left(S^2\right)$ in [11]. Ergün and Çalėkkan defined the concepts of the natural lift curve and geodesic spray in Minkowski 3-space and in Minkowski 4-space in [5,6].The anologue of the theorem of Thorpe was given in Minkowski 3-space and in Minkowski 4-space by Ergün and Çalėkkan in [4,5]. Çalėkkan and Ergün defined $\overline{M}$ -vector field Z, $\overline{M}$ -geodesic spray, $\overline{M}$ -integral curve of Z, $\overline{M}$ -geodesic in [13].The anologue of the theorem of Sivridaĝ and Çalėkkan was given in Minkowski 3-space by Ergün and Çalėkkan in [13] .At each point of a differentiable curve a tetrad of mutually orthogonal unit vectors (called tangent, normal,binormal, and trinormal) was defined and constructed. The rates of changes of these vectors along the curve

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define the curvatures of the curve in the space E_1^4 . Spherical images (indicatrices) are a well-known concept in classical differential geometry of curves [16].

To meet the requirements in the next sections, here the basic elements of the theory of curves in the space E_1^4 are briefly presented. (A more complete elementary treatment can be found in [10].)

Let Minkowski 4-space IR_1^4 be the vector space IR^4 equipped with the Lorentzian inner product g given by

$$g(X,X) = -x_1^2 + x_2^2 + x_3^2 + x_4^2$$

where $X = (x_1, x_2, x_3, x_4) \in \mathrm{IR}^4$. A non-zero vector $X = (x_1, x_2, x_3, x_4) \in \mathrm{IR}^4$ is said to be timelike if $g(X,X) < 0$, spacelike if $g(X,X) > 0$ and lightlike (or null) if $g(X,X) = 0$. The norm of a vector X is defined by

$$\|X\|_{IL} = \sqrt{|g(X,X)|}.[3]$$

We denote by $\{T(t), N(t), B_1(t), B_2(t)\}$ the moving Frenet frame along the curve α . Let α be a unit speed timelike space curve. The functions κ_1 , κ_2 and κ_3 are called the first, second and third curvature of α . Let Frenet vector fields of α be $\{T, N, B_1, B_2\}$. N is spacelike vector field. In this trihedron, we assume that T and B_1 are spacelike vector fields and B_2 is a timelike vector field .Then, Frenet formulas are given by

$$\dot{T} = \kappa_1 N, \quad \dot{N} = \kappa_1 T + \kappa_2 B_1, \; \dot{B_1} = -\kappa_2 N + \kappa_3 B_2, \; \dot{B_2} = -\kappa_3 B_1, \; [10].$$

Let α be any timelike curve with $\alpha = \alpha(t)$.

TypeI. General Case: $\kappa_1 \neq 0 \quad \kappa_2 \neq 0$ and $\kappa_3 = 0.$ The helix is not confined to any 3-flat in space-time [17].

TypeII. Degenerate: $\kappa_1 \neq 0 \quad \kappa_2 \neq 0$ and $\kappa_3 \neq 0.$ The helix is lies in a 3-flat [17].

Subtype $\mathbf{II_a}$: $\kappa_1^2 - \kappa_2^2 > 0$

Subtype $\mathbf{II_b}$: $\kappa_1^2 - \kappa_2^2 = 0$

Subtype $\mathbf{II_c}$: $\kappa_1^2 - \kappa_2^2 < 0.$

TypeIII. Degenerate: $\kappa_1 \neq 0 \quad \kappa_2 = 0$ and $\kappa_3 = 0.$ The helix is lies in a 2-flat : It is a pseudocircle, a curve of hyperbolic type,[17].

TypeIV. Degenerate: $\kappa_1 = 0 \quad \kappa_2 = 0$ and $\kappa_3 = 0$. The helix is a straight line, [17].
Let M be a hypersurface in IR_1^4 and let $\alpha : I \to M$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{dt}\left(\alpha\left(t\right)\right) = X\left(\alpha\left(t\right)\right) \left(\text{for all } t \in I\right)$$

where X is a smooth tangent vector field on M, [3]. We have

$$TM = \bigcup_{P \in M} T_P M = \chi\left(M\right)$$

where $T_P M$ is the tangent space of M at P and $\chi\left(M\right)$ is the space of vector fields of M .

Definition1. *For any parametrized curve* $\alpha : I \to M$, $\overline{\alpha} : I \to TM$ *given by*

$$\overline{\alpha}\left(t\right) = \left(\alpha\left(t\right), \dot{\alpha}\left(t\right)\right) = \dot{\alpha}\left(t\right)|_{\alpha(t)}$$

is called the natural lift of ૬ *on* TM,[6]. *Thus, we can write*

$$\frac{d\overline{\alpha}}{dt} = \frac{d}{dt}\left(\dot{\alpha}\left(t\right)|_{\alpha(t)}\right) = D_{\dot{\alpha}(t)}\, \dot{\alpha}\left(t\right)$$

where D *is the Levi-Civita connection on* IR_1^4 .

Let α be a unit speed timelike space curve . Then the natural lift $\overline{\alpha}$ of α is a spacelike space curve, [6].

Definition 2. *A* $X \in \chi\left(TM\right)$ *is called a geodesic spray if for* $V \in TM$

$$X\left(V\right) = \varepsilon g\left(S\left(V\right), V\right) N$$

where $\varepsilon = g\left(N, N\right)$, [6]

Theorem 1. *The natural lift* $\overline{\alpha}$ *of the curve* α *is an integral curve of geodesic spray* X *if and only if* α *is a geodesic on* M , [6].

2. THE NATURAL LIFT CURVE OF THE SPHERICAL INDICATRIX OF A TIMELIKE CURVE IN MINKOWSKI 4-SPACE

Let D , $\bar{D}$ and $\bar{\bar{D}}$ be connections in IR_1^4 , S_1^3 and H_0^3 respectively and ξ be a unit normal vector field of S_1^3 and H_0^3 . Then Gauss Equations are given by the followings

$$\begin{cases} D_X Y = \bar{D}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi, \\ D_X Y = \bar{\bar{D}}_X Y + \varepsilon g\left(S\left(X\right), Y\right)\xi, \end{cases}$$

where and $\varepsilon = g\left(\xi, \xi\right)$ and S is the shape operator of S_1^3 and H_0^3 and

$$S\left(X\right) = -D_X \xi.$$

Let α be a unit speed timelike space curve, T is timelike vector field, N , B_1 and B_2 are spacelike vector fields .

2.1. The Natural Lift of the Spherical Indicatrix of Tangent Vector of α

Let α_T be the spherical indicatrix of tangent vectors of α and $\overline{\alpha_T}$ be the natural lift of the curve α_T . If $\overline{\alpha_T}$ is an integral curve of thegeodesic spray, then from **Theorem 1.** we have

$$\bar{\bar{D}}_{\dot{\alpha}_T} \dot{\alpha}_T = 0$$

that is

$$D_{\dot{\alpha}_T} \dot{\alpha}_T = \bar{\bar{D}}_{\dot{\alpha}_T} \dot{\alpha}_T + \varepsilon g\left(S\left(\dot{\alpha}_T\right), \dot{\alpha}_T\right)\xi$$

$$D_{\dot{\alpha}_T} \dot{\alpha}_T = \varepsilon g\left(S\left(\dot{\alpha}_T\right), \dot{\alpha}_T\right)T$$

$$D_{\dot{\alpha}_T} \dot{\alpha}_T = D_{\dot{\alpha}_T}\left(\kappa_1 N\right) = \left(\dot{\kappa}_1 N + \kappa_1\left(\kappa_1 T + \kappa_2 B_1\right)\right)\frac{ds}{ds_T},$$

$$\frac{ds}{ds_T} = \frac{1}{\kappa_1}$$

$$D_{\dot{\alpha}_T} \dot{\alpha}_T = \left(\dot{\kappa}_1 N + \kappa_1\left(\kappa_1 T + \kappa_2 B_1\right)\right)\frac{1}{\kappa_1}$$

and

$$\varepsilon = g\left(\xi, \xi\right) = g\left(T, T\right) = -1, g\left(S\left(\dot{\alpha}_T\right), \dot{\alpha}_T\right) = -\kappa_1^2.$$

Using these in the Gauss equation, we immediately have

$$\bar{\bar{D}}_{\dot{\alpha}_T}\,\dot{\alpha}_T=\left(\dot{\kappa}_1 N+\kappa_1\left(\kappa_1 T+\kappa_2 B_1\right)\right)\frac{1}{\kappa_1}-\kappa_1^2 T$$

From the Eq. (2.3) we get

$$\left(\frac{\kappa_1^2}{\kappa_1}-\kappa_1^2\right)T+\left(\frac{\dot{\kappa}_1}{\kappa_1}\right)N+\left(\frac{\kappa_1\kappa_2}{\kappa_1}\right)B_1=0$$

Since T, N, B_1, B_2 are linearly independent we obtain

$$\begin{cases}\kappa_1-\kappa_1^2=0\\ \frac{\dot{\kappa}_1}{\kappa_1}=0\\ \frac{\kappa_1\kappa_2}{\kappa_1}=0\end{cases}$$

So from the Eq. (2.7) we can give the following proposition.

Corollary 1. *If the natural lift $\overline{\alpha_T}$ of α_T is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$,then α can be helices of Type III.*

2.2. The Natural Lift of the Spherical Indicatrix of Principal Normal Vector of α

Let α_N be the spherical indicatrix of tangent vectors of α and $\overline{\alpha_N}$ be the natural lift of the curve α_N . If $\overline{\alpha_N}$ is an integral curve of the geodesic spray, then because of **Theorem 1.** we have

$$\bar{D}_{\dot{\alpha}_N}\,\dot{\alpha}_N=0$$

that is

$$D_{\dot{\alpha}_N}\,\dot{\alpha}_N=\bar{D}_{\dot{\alpha}_N}\,\dot{\alpha}_N+\varepsilon g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right)\xi$$

$$D_{\dot{\alpha}_N}\,\dot{\alpha}_N=\varepsilon g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right)N$$

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = D_{\dot{\alpha}_N}\left(\kappa_1 T + \kappa_2 B_1\right)$$
$$= \left(\dot{\kappa}_1 T + \kappa_1\left(\kappa_1 N\right) + \dot{\kappa}_2 B_1 + \kappa_2\left(-\kappa_2 N + \kappa_3 B_2\right)\right)\frac{ds}{ds_N},$$
$$\frac{ds}{ds_N} = \frac{1}{\sqrt{\left(-\kappa_1^2 + \kappa_2^2\right)}}$$

$$D_{\dot{\alpha}_N}\dot{\alpha}_N = \left(\dot{\kappa}_1 T + \kappa_1\left(\kappa_1 N\right) + \dot{\kappa}_2 B_1 + \kappa_2\left(-\kappa_2 N + \kappa_3 B_2\right)\right)\frac{1}{\sqrt{\left(-\kappa_1^2 + \kappa_2^2\right)}}$$

and

$$\varepsilon = g\left(\xi,\xi\right) = g\left(N,N\right) = 1, g\left(S\left(\dot{\alpha}_N\right),\dot{\alpha}_N\right) = \kappa_1^2 - \kappa_2^2.$$

Using these in the Gauss equation, we immediately have

$$\bar{D}_{\dot{\alpha}_N}\dot{\alpha}_N = \left(\dot{\kappa}_1 T + \left(\kappa_1^2 - \kappa_2^2\right)N + \kappa_3\kappa_2 B_2 + \dot{\kappa}_2 B_1\right)\frac{1}{\sqrt{\left(-\kappa_1^2 + \kappa_2^2\right)}}$$
$$+\left(-\kappa_1^2 + \kappa_2^2\right)N$$

From the Eq. (2.8) we get

$$\left(\frac{\dot{\kappa}_1}{\sqrt{\left(-\kappa_1^2 + \kappa_2^2\right)}}\right)T + \left(\frac{\kappa_1^2 - \kappa_2^2}{\sqrt{\left(-\kappa_1^2 + \kappa_2^2\right)}} + \left(-\kappa_1^2 + \kappa_2^2\right)\right)N$$
$$+\left(\frac{\dot{\kappa}_2}{\sqrt{\left(-\kappa_1^2 + \kappa_2^2\right)}}\right)B_1 + \left(\frac{\kappa_3\kappa_2}{\sqrt{\left(-\kappa_1^2 + \kappa_2^2\right)}}\right)B_2 = 0$$

Since T, N, B_1, B_2 are linearly independent we obtain ,

$$\begin{cases} \dfrac{\dot{\kappa}_1}{\sqrt{\left(-\kappa_1^2+\kappa_2^2\right)}} = 0 \\ \dfrac{\kappa_1^2-\kappa_2^2}{\sqrt{\left(-\kappa_1^2+\kappa_2^2\right)}} + \left(-\kappa_1^2 + \kappa_2^2\right) = 0 \\ \dfrac{\dot{\kappa}_2}{\sqrt{\left(-\kappa_1^2+\kappa_2^2\right)}} = 0 \\ \dfrac{\kappa_3\kappa_2}{\sqrt{\left(-\kappa_1^2+\kappa_2^2\right)}} = 0 \end{cases}$$

So from the Eq. (2.12) we can give the following proposition.

Corollary 2. *If the natural lift* $\overline{\alpha}_N$ *of* α_N *is an integral curve of the geodesic on the tangent bundle* $T\left(S_1^3\right)$ *,then* α *can be helices of Type II (Subtype II* $_a$ *or Subtype II* $_c$ *).*

2.3. The Natural Lift of the Spherical Indicatrix of the Firsth Binormal Vectors of α

Let α_{B_1} be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_{B_1}$ be the natural lift of the curve α_{B_1} . If $\overline{\alpha}_{B_1}$ is an integral curve of the geodesic spray, then by using **Theorem 1.** we have

$$\bar{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=0$$

that is

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\bar{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}+\varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)\xi$$

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\varepsilon g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)B_1$$

$$\begin{aligned}D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}&=D_{\dot{\alpha}_{B_1}}\left(-\kappa_2N+\kappa_3B_2\right)\\&=\left(-\dot{\kappa}_2N-\kappa_2\left(\kappa_1T+\kappa_2B_1\right)+\dot{\kappa}_3B_2+\kappa_3\left(-\kappa_3B_1\right)\right)\frac{ds}{ds_{B_1}},\end{aligned}$$

$$\frac{ds}{ds_{B_1}}=\frac{1}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}$$

$$D_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}=\left(-\kappa_1\kappa_2T-\dot{\kappa}_2N+\left(-\kappa_2^2-\kappa_3^2\right)B_1+\dot{\kappa}_3B_2\right)\frac{1}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}$$

and

$$\varepsilon=g\left(\xi,\xi\right)=g\left(B_1,B_1\right)=1,g\left(S\left(\dot{\alpha}_{B_1}\right),\dot{\alpha}_{B_1}\right)=-\left(\kappa_2^2+\kappa_3^2\right).$$

Using these in the Gauss equation, we immediately have

$$\begin{aligned}\bar{D}_{\dot{\alpha}_{B_1}}\dot{\alpha}_{B_1}&=\left(-\kappa_1\kappa_2T-\dot{\kappa}_2N+\left(-\kappa_2^2-\kappa_3^2\right)B_1+\dot{\kappa}_3B_2\right)\frac{1}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}\\&\quad+\left(\kappa_2^2+\kappa_3^2\right)B_1\end{aligned}$$

From the Eq. (2.13) we get

$$\left(\frac{\kappa_1\kappa_2}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}\right)T+\left(\frac{\dot{\kappa}_2}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}\right)N$$

$$+\left(-\frac{\kappa_2^2+\kappa_3^2}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}+\left(\kappa_2^2+\kappa_3^2\right)\right)B_1+\left(\frac{\dot{\kappa}_3}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}\right)B_2=0$$

Because T, N, B_1, B_2 are linearly independent, we have ,

$$\begin{cases} \dfrac{\kappa_1\kappa_2}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}=0 \\ \dfrac{\dot{\kappa}_2}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}=0 \\ -\dfrac{\kappa_2^2+\kappa_3^2}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}+\left(\kappa_2^2+\kappa_3^2\right)=0 \\ \dfrac{\dot{\kappa}_3}{\sqrt{\left(\kappa_2^2+\kappa_3^2\right)}}=0 \end{cases}$$

So from the Eq. (2.17) we can give the following proposition.

Corollary 3. *If the natural lift $\overline{\alpha}_{B_1}$ of α_{B_1} is an integral curve of the geodesic on the tangent bundle $T\left(S_1^3\right)$,then α can be helices of Type II or helices of Type III.*

2.4. The Natural Lift of the Spherical Indicatrix of the Second Binormal Vectors of α

Let α_{B_2} be the spherical indicatrix of tangent vectors of α and $\overline{\alpha}_{B_2}$ be the natural lift of the curve α_{B_1} . If $\overline{\alpha}_{B_2}$ is an integral curve of the geodesic spray, then by using **Theorem 1.** we have

$$\bar{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=0$$

that is

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\bar{D}_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}+\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)\xi$$

$$D_{\dot{\alpha}_{B_2}}\dot{\alpha}_{B_2}=\varepsilon g\left(S\left(\dot{\alpha}_{B_2}\right),\dot{\alpha}_{B_2}\right)B_2$$

$$D_{\dot{\alpha}_{B_2}} \dot{\alpha}_{B_2} = D_{\dot{\alpha}_{B_2}} \left(-\kappa_3 B_1\right)$$
$$= \left(-\dot{\kappa}_3 B_1 - -\kappa_3\left(-\kappa_2 N + \kappa_3 B_2\right)\right)\frac{ds}{ds_{B_2}},$$
$$\frac{ds}{ds_{B_2}} = \frac{1}{\kappa_3}$$

$$D_{\dot{\alpha}_{B_2}} \dot{\alpha}_{B_2} = \left(-\dot{\kappa}_3 B_1 - -\kappa_3\left(-\kappa_2 N + \kappa_3 B_2\right)\right)\frac{1}{\kappa_3}$$

and

$$\varepsilon = g\left(\xi,\xi\right) = g\left(B_2, B_2\right) = 1, g\left(S\left(\dot{\alpha}_{B_2}\right), \dot{\alpha}_{B_2}\right) = \kappa_3^2.$$

Using these in the Gauss equation, we immediately have

$$\bar{D}_{\dot{\alpha}_{B_2}} \dot{\alpha}_{B_2} = \left(-\dot{\kappa}_3 B_1 - \kappa_3\kappa_2 N + \kappa_3^2 B_2\right)\frac{1}{\kappa_3} - \kappa_3^2 B_2$$

From the Eq. (2.18) we get

$$\left(-\frac{\kappa_3\kappa_2}{\kappa_3}\right)N + \left(-\frac{\dot{\kappa}_3}{\kappa_3}\right)B_1 + \left(\frac{\kappa_3^2}{\kappa_3} - \kappa_3^2\right)B_2 = 0$$

Because T, N, B_1, B_2 are linearly independent, we have,

$$\begin{cases} -\frac{\kappa_3\kappa_2}{\kappa_3} = 0 \\ -\frac{\dot{\kappa}_3}{\kappa_3} = 0 \\ \frac{\kappa_3^2}{\kappa_3} - \kappa_3^2 = 0 \end{cases}$$

So from the Eq. (2.22) we can give the following proposition.

Corollary 4. *If the natural lift $\overline{\alpha}_{B_2}$ of α_{B_2} is an integral curve of the geodesic on the tangent bundle $T\left(H_0^3\right)$ then we have $\kappa_2 = 0$ and $\kappa_3 = 1$. Therefore there is no such curve satisfying this condition.*

REFERENCES

[1] Sivridaĵ , A.Ķ, Çalēkkan, M., *Erc.Uni. Fen Bil. Derg*., **7**(2), 1283, 1991.
[2] O'Neill, B., *Elementery Differential Geometry,* Academic Press, New York and London, 1967.
[3] O'Neill, B., *Semi-Riemannian Geometry, with applications to relativity*, Academic Press, New York, 1983.
[4] Bonnor, W. B., *Tensor*, **20**, 229, 1969.
[5] Ergün, E., Çalēkkan, M., *International Journal of Contemp. Math. Sciences*, **6**(39), 1929, 2011.
[6] Ergün, E., Çalēkkan, M., *International Journal of Contemporary Mathematical Sciences*, **7**(12), 575, 2012.
[7] Ergün, E., Çalēkkan, M., *International Mathematical Forum*, **7**(15), 707, 2012.
[8] Ergün, E., Bilici, M., Çalēkkan, M., *Journal of Science and Arts*, **1**(30), 39, 2015.
[9] Thorpe, J.A., *Elementary Topics In Differential Geometry*, Springer-Verlag, New York, Heidelberg-Berlin, 1979.
[10] Walrave, J., *Curves and Surfaces in Minkowski Space*, K. U. Leuven Faculteit Der Wetenschappen, 1995.
[11] Bilici, M., Çalēkkan, M., Aydemir, Ķ, *Journal of Applied Mathematics*, **11**(4), 415, 2002.
[12] Çalēkkan, M., Sivridaĵ , A.Ķ, Hacēsalihoĵ lu, H.H., Some Characterizations for the natural lift curves and the geodesic spray, *Communications, Fac. Sci.Univ.Ankara Ser. AMath.*, **33**, 1984.
[13] Çalēkkan, Ergün, E., *Int. J. of Contemp. Math. Sciences*, **6**(39), 1935, 2011.
[14] Petroviç ï Torgasev, M., Sucuroviç, E., *Novi Sad J. Math*., **32**(2), 55, 2002.
[15] Do Carmo, M.P., *Differential Geometry of Curve and Surface*, Prentice-Hall Inc. Englewood Cliffs, New Jersey, 1976.
[16] Milman R. S., Parker G. D., *Elements of Differential Geometry*, Prentice-Hall Inc., Englewood Cliffs, New Jersey, 1977.
[17] Synge, J. L., *Proc. Roy. Irish Academy*, **A65**, 27, 1967.