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SOLVING SYSTEM OF SECOND ORDER BOUNDARY VALUE PROBLEMS BY NUMEROV TYPE METHOD

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Abstract. *The aim of this paper is to describe an efficient numerical method for solving system of two-point boundary value problems subject to the Dirichelet boundary conditions. We present the construction of Numerov type method by assuming extra continuity condition on the solution. The order of the propose method is quadratic. This propose method is suitable for solving an obstacle problems. We have considered an obstacle problem and solved by the propose method to illustrate the efficiency and the accuracy. Results are compared with the other methods.*

Keywords: *Boundary value problems, Finite difference method, Obstacle problem, Quadratic order method, Second order system of differential equation.*
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1. INTRODUCTION

In this work, we consider a system of the second order boundary value problems of the form

$$u''(x) = \begin{cases} f_1(x,u) & a \le x \le c \\ f_2(x,u) & c \le x \le d \\ f_3(x,u) & d \le x \le b \end{cases} \tag{1}$$

with the Dirichelet boundary conditions

$$y(a) = \alpha\ , \ \ y(b) = \beta$$

where α, β are real finite constants and real functions $f_1(x,u)$, $f_2(x,u)$, $f_3(x,u)$ are respectively continuous in $[a,c]$, $[c,d]$, $[d,b]$. Also $u(x)$, $u'(x)$ are continuous at c and d.

An ordinary differential equations are used to model different kind of problems in all branches of engineering and sciences. Generally such type of systems arise in modeling and the study of one dimensional obstacle, unilateral, moving and free boundary value problems, [1- 4] and the references therein. In most cases it is impossible to obtain solutions of these problems for arbitrary choice of source function $f_i(x,u), i = 1,2,3$ using analytical methods which satisfy the given boundary conditions. So in these cases we resort on approximate solution of the problems. A literature regarding the numerical solution of the two-point boundary value problem is given in [5-8]. The existence and uniqueness of the solution for the problem (1) is assumed. The specific assumption on $f_i(x,y),\ i = 1,2,3$ to ensure existence and uniqueness will not be considered [1, 5, 9] in this article. In specific problems (1), there are many different methods and approaches such as collocation [10], splines [11], finite difference method [12, 13] , finite element [14] that are used to derive the approximate solutions.

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In this article we shall develop a Numerov type finite difference method for solving problems (1) numerically. The accuracy of the propose method is at least quadratic. A numerical comparison between the present and other methods reported in literature, to demonstrate the effectiveness of the method is given.

We have presented our work in this article as follows. In the next section we will derive finite difference method. In Section 3, we have discussed local truncation error in propose method and convergence under appropriate condition in Section 4. The applications of the proposed method to the model problems and illustrative numerical results have been produced to show the efficiency in Section 5. Discussion and conclusion on the performance of the method are presented in Section 6.

2. DEVELOPMENT AND DERIVATION OF THE METHOD

We define N finite numbers of nodal points of the domain [a, b], in which the solution of the problem (1) is desired, as $a = x_0 < x_1 \ldots\ldots\ldots < x_{N-1} < x_N = b$ using uniform step length h such that $x_i = a + i.h, i = 0\ (1)\ N$. Let denote the exact solution $u(x)$ at $x = x_i$ by u_i. Also let us denote f_{ji} as the approximation of the theoretical value of the force function $f_j(x, \mathrm{u}), \mathrm{j} = 1,2,3$ at node $x = x_i,\ i = 1\ (1)\ N$. We can define other notations used in this article i.e. $f_{ji\pm1}$ and $u_{i\pm1}$in the similar way. Also we define $x_{i+\frac{1}{2}} = x_i + \frac{h}{2}$ and $u''_{i+\frac{1}{2}} = f_j\left(x_{i+\frac{1}{2}}, u_{i+\frac{1}{2}}\right) = f_{ji+\frac{1}{2}}$ …etc for $i = 0\ (1)\ N-1$. Suppose we have to determine a number u_i, which is an approximation to the numerical value of the theoretical solution $u(x)$ of the problem (1) at the nodal point $\mathrm{x} = x_i, i = 1\ (1)\ N-1$. Thus the problem (1) at node x_i may be written as,

$$u''_i = \begin{cases} f_{1i} & a \le x = x_i \le c \\ f_{2i} & c \le x = x_i \le d \\ f_{3i} & d \le x = x_i \le b \end{cases} \tag{2}$$

Using method of undetermined coefficients, Taylor's series expansion and following the ideas in [15, 16] we descretize problem (2) at nodes in [a, b],

$$\begin{aligned} 3u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= 2u_{i-1} + \frac{h^2}{4}\left(u''_{i-1} - 4u''_{i-\frac{1}{2}}\right) + r_i\,, & i &= 1, \\ -u_{i-\frac{3}{2}} + 2u_{i-\frac{1}{2}} - u_{i+\frac{1}{2}} &= -h^2 u''_{i-\frac{1}{2}} + r_i\,, & 2 &\le i \le N-1, \\ -u_{i-\frac{3}{2}} + 3u_{i-\frac{1}{2}} &= 2u_i + \frac{h^2}{4}\left(-4u''_{i-\frac{1}{2}} + u''_i\right) + r_i\,, & i &= N\,, \end{aligned} \tag{3}$$

Thus, after neglecting the remainder terms $r_i\,, i = 1,2, \ldots., N$ in (3), we will obtain our propose Numerov Type finite difference method as a system of equations in $u_{i-\frac{1}{2}}$, $i = 1,2, \ldots\ldots, N$.

Thus, the method consists in finding an approximation u_i for the theoretical solution $u(x_i)$, $i = 1(1)N-1$ of the problem (1) by solving the system $(N-1) \times (N-1)$ equations (3) in $u_{i-\frac{1}{2}}$. Finally we approximate u_i by second order difference approximation

$$u_i = \frac{1}{2}\left(u_{i+\frac{1}{2}} + u_{i-\frac{1}{2}}\right), \qquad i = 1,2, \ldots\ldots\ldots, N-1. \tag{4}$$

3. THE CONVERGENCE ANALYSIS OF THE METHOD

In this section we consider the convergence of the method (3). So we consider following linear problem,

$$0 = -u''(x) + \begin{cases} f_1(x) & a \le x \le c \\ f_2(x) & c \le x \le d \\ f_3(x) & d \le x \le b \end{cases} \tag{5}$$

subject to the boundary conditions $u(a) = \alpha$ and $u(b) = \beta$. We can write the proposed method (3) in the matrix form

$$\boldsymbol{DU} + \boldsymbol{S} + \boldsymbol{R} = \boldsymbol{0} \tag{6}$$

where $\boldsymbol{D} = (d_{lm})_{N\times N}$ a tri-diagonal matrix and defined as

$$d_{lm} = \begin{cases} 3 & l = m, \quad l = 1, N \\ 2 & 2 \le l = m \le N-1 \\ -1 & |l-m| = 1 \end{cases} \tag{7}$$

$\boldsymbol{U} = (U\left(x_{i-\frac{1}{2}}\right))_{N\times 1}, 1 \le l = i \le N$ is exact solution of problem (5), $\boldsymbol{S} = (s_l)$ and $\boldsymbol{R} = (r_l)$ are N-dimensional column vectors such that

$$s_l = \begin{cases} -2\alpha - \dfrac{h^2}{4}\left(f_1(x_{i-1}) - 4f_1(x_{i-\frac{1}{2}})\right), & l = i = 1 \\ h^2 f_j\left(x_{i-\frac{1}{2}}\right), & 2 \le l = i \le N-1, \ \ j = 1,2,3 \\ -2\beta - \dfrac{h^2}{4}\left(f_3(x_i) - 4f_3(x_{i-\frac{1}{2}})\right), & l = i = N \end{cases} \tag{8}$$

and

$$r_l = \begin{cases} -\dfrac{h^4}{64}u^{(4)}\left(x_{i-\frac{1}{2}} + \theta_i h\right), & 0 \le \theta_i \le 1, \quad l = i = 1 \\ \dfrac{h^4}{12}u^{(4)}\left(x_{i-\frac{1}{2}} + \theta_i h\right), & 2 \le l = i \le N-1, \\ -\dfrac{h^4}{64}u^{(4)}\left(x_{i-\frac{1}{2}} + \theta_i h\right), & l = i = N \end{cases} \tag{9}$$

Let $u_{i-\frac{1}{2}}$ be an approximate value of $U_{i-\frac{1}{2}}, i = 1,2, \dots \dots, N$ which we obtained by solving test problem (5) by propose method (3) after neglecting the remainder terms. We write (3) in matrix form,

$$\boldsymbol{Du} + \boldsymbol{S} = \boldsymbol{0} \tag{10}$$

where $\boldsymbol{u} = (u\left(x_{i-\frac{1}{2}}\right))_{N\times 1}, 1 \le l = i \le N$ is approximate solution of problem (5). From (6) and (10) we have,

$$\boldsymbol{D}(\boldsymbol{U} - \boldsymbol{u}) + \boldsymbol{R} = \boldsymbol{0} \tag{11}$$

Let define an error in exact and approximate solution of (5), $e_{i-\frac{1}{2}} = U_{i-\frac{1}{2}} - u_{i-\frac{1}{2}}$.

Thus from (11) we have,

$$\boldsymbol{De} + \boldsymbol{R} = \boldsymbol{0} \tag{12}$$

where $\boldsymbol{e} = (e_{i-\frac{1}{2}})_{N\times 1}, 1 \le l = i \le N$. Thus from (12), we conclude that the convergence of the propose difference method depends on the property of matrix $\boldsymbol{D}$. To simplify we need to determine $\boldsymbol{D^{-1}}$, by the row sum criterion $\boldsymbol{D}$ is monotone [17]. Thus $\boldsymbol{D^{-1}}$ exist and $\boldsymbol{D^{-1}} \geq 0$. We determine $\boldsymbol{D^{-1}} = (d_{lm}^{-1})$ explicitly where

$$d_{lm}^{-1} = \begin{cases} \dfrac{(2l-1)(2N-2m+1)}{4N} & l \leq m \\ \dfrac{(2m-1)(2N-2l+1)}{4N} & l \geq m \end{cases} \tag{13}$$

and the row sum of $\boldsymbol{D^{-1}}$ is

$$\sum_{m=1}^{N} d_{lm}^{-1} = \frac{2l(N-l)-(N-2l)}{4} \tag{14}$$

Hence from (13) we obtain

$$\|\boldsymbol{D^{-1}}\| = \max_{1\leq l\leq N} \sum_{m=1}^{N} |d_{lm}^{-1}| \leq \frac{1}{2}\left(\frac{(b-a)^2}{h^2}+1\right) \tag{15}$$

Thus from equation (12) and (15), we have

$$\|\boldsymbol{e}\| \leq \frac{1}{2}\left(\frac{(b-a)^2}{h^2}+1\right)\|\boldsymbol{R}\| \tag{16}$$

Let $M = \max_{a\leq x\leq b} \left|u^{(4)}(x)\right|$, then from (9) and (16) we have

$$\|\boldsymbol{e}\| \leq \frac{h^4}{24}\left(\frac{(b-a)^2}{h^2}+1\right)M \leq O(h^2) \tag{17}$$

Thus we conclude from equation (17) that $\|\boldsymbol{e}\| \to 0$ as $h \to 0$. Thus we have proved that theoretically propose method is convergent and order of convergence is at least quadratic.

4. NUMERICAL EXPERIMENTS

In this section, we have applied the proposed method (3) to solve numerically two different model problems. We have used Gauss-Seidel iteration method to solve the system of linear equations arises from equation (3). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. Let u_i , the numerical value calculated by formulae (3), an approximate value of the theoretical solution $u(x)$ at the grid point $x = x_i$. The maximum absolute error

$$MAE(\mathrm{u}) = \max_{1\leq i\leq N-1} |u(x_i) - u_i|$$

are shown in Tables 1-6, for different value of h, the step length. Also we have shown different values of *MAE* in the tables reported in literature for comparison purpose computed by different method.

Example 1. Consider the following linear system of two-point boundary value problem

$$u''(x) = \begin{cases} 0 & 0 \le x \le \frac{\pi}{4} \\ u(x) - 1 & \frac{\pi}{4} \le x \le \frac{3\pi}{4} \\ 0 & \frac{3\pi}{4} \le x \le \pi \end{cases}$$

with the boundary conditions $y(0) = 0,\ y(\pi) = 0$. In Tables1-4, the maximum absolute error presented in exact solution

$$u(x) = \begin{cases} \frac{4}{\gamma_1} x & 0 \le x \le \frac{\pi}{4} \\ 1 - \frac{4}{\gamma_2} \cosh\left(\frac{\pi}{2} - x\right) & \frac{\pi}{4} \le x \le \frac{3\pi}{4} \\ \frac{4}{\gamma_1} (\pi - x\ ) & \frac{3\pi}{4} \le x \le \pi \end{cases}$$

where $\gamma_1 = \pi + 4\coth\left(\frac{\pi}{4}\right)$ and $\gamma_2 = \pi \sinh\left(\frac{\pi}{4}\right) + 4\cosh\left(\frac{\pi}{4}\right)$.

Example 2. In [21], consider the following linear system of two-point boundary value problem

$$u''(x) = \begin{cases} 2 & 0 \le x \le \frac{\pi}{4} \\ u(x) + 1 & \frac{\pi}{4} \le x \le \frac{3\pi}{4} \\ 2 & \frac{3\pi}{4} \le x \le \pi \end{cases}$$

with the boundary conditions $y(0) = 0,\ y(\pi) = 0$. In Tables 5-6, the the maximum absolute error presented in exact solution

$$u(x) = \begin{cases} x^2 + \left(\frac{(\pi^2 - 16)\sinh\left(\frac{\pi}{4}\right)}{4\gamma} - \frac{\pi}{2}\right) x & 0 \le x \le \frac{\pi}{4} \\ -1 - \frac{(\pi^2 - 16)}{4\gamma} \cosh\left(\frac{\pi}{2} - x\right) & \frac{\pi}{4} \le x \le \frac{3\pi}{4} \\ x^2 + \left(\frac{(\pi^2 - 16)\sinh\left(\frac{\pi}{4}\right)}{4\gamma} + \frac{3\pi}{2}\right)\left(\frac{3\pi}{4} - x\right) - \left(\frac{9\pi^2}{16} + \frac{(\pi^2 - 16)}{4\gamma}\cosh\left(\frac{\pi}{4}\right) + 1\right) & \frac{3\pi}{4} \le x \le \pi \end{cases}$$

where $\gamma = \pi \sinh\left(\frac{\pi}{4}\right) + 4\cosh\left(\frac{\pi}{4}\right)$.

Table 1. Maximum absolute error $\left|u(x_i) - u_i\right|$ in example 1.

N	*MAE*			*MAE*
	$0 \le x \le \frac{\pi}{4}$	$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$	$\frac{3\pi}{4} \le x \le \pi$	
16	.20767539(-2)	.28324185(-2)	.15575651(-2)	.28324185(-2)
32	.64796966(-3)	.74993078(-3)	.39345660(-3)	.74993078(-3)
64	.17962954(-3)	.19265356(-3)	.98950048(-4)	.19265356(-3)
128	.47008900(-4)	.48952381(-4)	.24801866(-4)	.48952381(-4)
256	.11527084(-4)	.12338443(-4)	.62350191(-5)	.12338443(-4)

512	.26414673(-5)	.31437369(-5)	.15560545(-5)	.31437369(-5)
1024	.62554716(-6)	.81915579(-6)	.39376394(-6)	.81915579(-6)
2048	.13912775(-6)	.25000776(-6)	.95740724(-7)	.25000776(-6)

N	MAE			MAE
	$0 \le x \le \frac{\pi}{4}$	$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$	$\frac{3\pi}{4} \le x \le \pi$	
20	.14558636(-2)	.18513107(-2)	.10009173(-2)	.18513107(-2)
40	.43250457(-3)	.48430692(-3)	.25231281(-3)	.48430692(-3)
80	.11727104(-3)	.12406982(-3)	.63366075(-4)	.12406982(-3)
160	.30224175(-4)	.31470874(-4)	.15920774(-4)	.31470874(-4)
320	.72047378(-5)	.78972371(-5)	.39998449(-5)	.78972371(-5)

Table 2. Comparison of the maximum absolute error $\left|u\left(x_i\right)-u_i\right|$ in example 1.

N	MAE	
	Our method	[13]
32	.74993078(-3)	.1183(-2)
64	.19265356(-3)	.3032(-3)
128	.48952381(-4)	.6892(-4)

N	MAE			
	Our method	[13]	[18]	[19]
20	.18513107(-2)	.165(-2)	.220(-2)	.194(-2)
40	.48430692(-3)	.433(-3)	.587(-3)	.499(-3)
80	.12406982(-3)	.111(-3)	.151(-3)	.127(-3)

N	MAE			
	scheme (20) [12]	[12]	[20]	[10]
20	.250(-1)	.232(-1)	.182(-1)	.140(-1)
40	.129(-1)	.121(-1)	.917(-2)	.771(-2)
80	.658(-2)	.617(-2)	.461(-2)	.404(-2)

Table 3. Maximum absolute error $\left|u\left(x_{(i-1/2)}\right)-u_{(i-1/2)}\right|$ in example 1.

N	MAE			MAE
	$0 \le x \le \frac{\pi}{4}$	$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$	$\frac{3\pi}{4} \le x \le \pi$	
16	.24228767(-2)	.11897720(-7)	.75432469(-8)	.24228767(-2)
32	.69425604(-3)	.11598400(-7)	.14858972(-7)	.69425604(-3)
64	.18564930(-3)	.22813627(-7)	.11848045(-7)	.18564930(-3)
128	.47885918(-4)	.23470047(-7)	.12605618(-7)	.47885918(-4)
256	.11820583(-4)	.27208914(-7)	.14860534(-7)	.11820583(-4)
512	.28011168(-5)	.27879866(-7)	.14716609(-7)	.28011168(-5)
1024	.70537186(-6)	.28689437(-7)	.14870607(-7)	.70537186(-6)
2048	.16413892(-6)	.29536370(-7)	.14599374(-7)	.16413892(-6)

Table 4. Comparison of the maximum absolute error $\left|u\left(x_{(i-1/2)}\right)-u_{(i-1/2)}\right|$ in example 1.

N	MAE	
	Our method	Eisa Al Said
32	.69425604(-3)	.9041(-3)
64	.18564930(-3)	.2350(-3)
128	.47885918(-4)	.5989(-4)

Table 5. Maximum absolute error $|u(x_i) - u_i|$ in example 2 .

N	MAE			MAE
	$0 \le x \le \frac{\pi}{4}$	$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$	$\frac{3\pi}{4} \le x \le \pi$	
16	.77617411(-2)	.21318281(-2)	.96382825(-2)	.77617411(-2)
32	.21615957(-2)	.42546634(-3)	.24097555(-2)	.24097555(-2)
64	.57064334(-3)	.91799520(-4)	.60254638(-3)	.60254638(-3)
128	.14698156(-3)	.21406437(-4)	.15077036(-3)	.15077036(-3)
256	.37537640(-4)	.51939742(-5)	.37845151(-4)	.37845151(-4)
512	.94746847(-5)	.14388813(-5)	.95309624(-5)	.95309624(-5)
1024	.24413368(-5)	.48520707(-6)	.25048184(-5)	.95309624(-5)
2048	.79235326(-6)	.13631579(-6)	.75771072(-6)	.79235326(-6)

N	MAE			MAE
	$0 \le x \le \frac{\pi}{4}$	$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$	$\frac{3\pi}{4} \le x \le \pi$	
20	.51864525(-2)	.12580240(-2)	.61685122(-2)	.61685122(-2)
40	.14137383(-2)	.25761966(-3)	.15421555(-2)	.15421555(-2)
80	.36917866(-3)	.56751993(-4)	.38558699(-3)	.38558699(-3)
160	.94546485(-4)	.13359811(-4)	.96587333(-4)	.96587333(-4)
320	.24005367(-4)	.32866253(-5)	.24266303(-4)	.24266303(-4)

Table 6. Maximum absolute error $\left|u\left(x_{i-\frac{1}{2}}\right) - u_{i-\frac{1}{2}}\right|$ in example 2 .

N	MAE			MAE
	$0 \le x \le \frac{\pi}{4}$	$\frac{\pi}{4} \le x \le \frac{3\pi}{4}$	$\frac{3\pi}{4} \le x \le \pi$	
16	.65680565(-2)	.29460409(-7)	.15409878(-7)	.65680565(-2)
32	.18600447(-2)	.24461363(-7)	.24997565(-7)	.18600447(-2)
64	.49444858(-3)	.26573861(-7)	.29476160(-7)	.49444858(-3)
128	.12676643(-3)	.29039197(-7)	.29102319(-7)	.12676643(-3)
256	.31954012(-4)	.29499715(-7)	.29360178(-7)	.31954012(-4)
512	.77428585(-5)	.29684921(-7)	.29785877(-7)	.77428585(-5)
1024	.16237202(-5)	.29742679(-7)	.29584607(-7)	.16237202(-5)
2048	.32899888(-6)	.29685774(-7)	.29575483(-7)	.32899888(-6)

CONCLUSION

A finite difference scheme is presented for numerical solution of system of two-point boundary value problems. It follows from derivation and discussion, the method (3) is at least of quadratic order which is well evident in computational results. We obtain comparable results using less number of functions evaluations at interior grid points unlike [13]. We can claim, in general that our method is better than other finite difference method. Numerical results show that our method generates results more accurate than that method in [13]. It is an alternative method, to get reliable results with less effort and obtain competitive results to those obtained with other methods with less computational cost.

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