*ORIGINAL PAPER*

# SOME PROPERTIES OF THE GENERALIZED COMPLETE AND INCOMPLETE BETA FUNCTIONS

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***Abstract.*** *In this paper, we derive certain relations between the integral of the incomplete generalized beta function and the complete version, which will also hold for the standard beta function. These results lead to relations for the generalized beta function and hence the standard beta functions. In particular, it is showed that the difference between the function with first variable shifted by any integer and that of the function with the first variable shifted by one is the same as the corresponding difference for the second variable.*

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## 1. INTRODUCTION

The beta function (also called Euler integral of the first kind) introduced by “Euler” plays an important role in the study of some important special functions. For details we refer to [2, 3, 9- 12]. This function can be written in terms of the gamma function as (Earl [9])

$$B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)} \tag{1.1}$$

The integral representation is given by [9, p. 18 (1)]

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt \quad (\Re(x)>0,\Re(y)>0) \tag{1.2}$$

The gamma functions, themselves can be decomposed into the two incomplete gamma functions [6]:

$$\gamma(\alpha,x)=\int_0^x t^{\alpha-1}\exp(-t)\,dt \quad (\Re(\alpha)>0) \tag{1.3}$$

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$$\Gamma(\alpha,x)=\int_x^\infty t^{\alpha-1}\exp(-t)\,dt \quad (\Re(\alpha)>0) \tag{1.4}$$

which led to the definition of the incomplete beta function

$$B_t(x,y)=\int_0^t h^{x-1}(1-h)^{y-1}\,dh \qquad (\Re(x)>0,\Re(y)>0 \ and \ 0\le t\le 1) \tag{1.5}$$

Recently, Chaudhry et al. [4] extended beta function by introducing a new parameter p and gave Euler-type integral representation as:

$$B(x,y,p)=\int_0^1 t^{x-1}(1-t)^{y-1}\exp\left[\frac{-p}{t(1-t)}\right]dt \qquad (\Re(p)>0) \tag{1.6}$$

More recently, Ozarslan et al. [8] considered further generalization of the extended beta function as

$$B^{(\alpha,\beta)}(x,y,p)=\int_0^1 t^{x-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt \qquad (\Re(p)>0). \tag{1.7}$$

For α = β, (1.7) reduces to (1.6). Clearly when p = 0, it reduces to the classical beta function. The generalization caries through trivially to the incomplete beta function. For more details we refer to Chaudhry et al. [4] and Chaudhry and Zubair [6]. As an example where such an extension was applied, is the extension of the Gauss hypergeometric function defined as [5]:

$$F_p(a,b,c,z)=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\sum_{n=0}^\infty \frac{(a)_n\,B(b+n,c-b;p)\,z^n}{n!} \tag{1.8}$$

$$\left(p\ge 0;|z|<1;\Re(c)>\Re(b)>0\right)$$

In this paper, we derive further properties relating these generalized beta function to the integral of its incomplete version. These relations lead to relations satisfied by the generalized beta function, including the relationship that the difference between $B^{(\alpha,\beta)}(x+n,y;p)$ and $B^{(\alpha,\beta)}(x+1,y;p)$ is the same as the corresponding difference when the shift is applied in the second variable. This result holds even for the standard beta function.

The plan of this paper is as follows: In Section 2, we give some general formulae that involve the incomplete generalized beta function. Then, by taking special cases of the general forms, some important and new relations are given. Also, a generalization of the well known functional property of the standard and generalized beta function is given. In Section 3, we apply results of Section 2 where new relations are found. One relation is between the complete version of the generalized beta function. The other is an elegant relationship that justifies the difference between two generalized beta functions with any integral shift in one of the variables. Further, in section 4, conclusion and remarks of this paper is presented.

## 2. INTEGRALS INVOLVING THE INCOMPLETE GENERALIZED BETA FUNCTION

As the independent variable of the incomplete extended beta function $t \in [0,1]$, the integrals of these functions are found over the unit interval. The following theorem shows that they can be expressed in closed forms.

**Theorem 1**. The following integral representation for generalized Beta function holds true:

$$\int_0^1 t^{s-1} B_t^{(\alpha,\beta)}(x,y;p)\,dt = \frac{1}{s}\left[ B^{(\alpha,\beta)}(x,y;p) - B^{(\alpha,\beta)}(x+s,y;p) \right] \quad (s \neq 0) \tag{2.1}$$

*Proof.* We know that

$$B^{(\alpha,\beta)}(x+s,y;p) = \int_0^1 t^{x+s-1}(1-t)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{t(1-t)}\right)dt$$

Taking $t^s$ as u and the remaining part as dv, we get

$$v = \int_0^t h^{x-1}(1-h)^{y-1}\,{}_1F_1\left(\alpha;\beta;\frac{-p}{h(1-h)}\right)dh$$

which is clearly $B_t^{(\alpha,\beta)}(x,y;p)$. Hence we have

$$B^{(\alpha,\beta)}(x+s,y;p) = B^{(\alpha,\beta)}(x,y;p) - s\int_0^1 t^{s-1} B_t^{(\alpha,\beta)}(x,y;p)\,dt \tag{2.2}$$

By rearranging the terms we get the result.

**Corollary 1.** Taking s = 1 in Theorem 1 we get the following result:

$$\int_0^1 B_t^{(\alpha,\beta)}(x,y;p)\,dt = B^{(\alpha,\beta)}(x,y;p) - B^{(\alpha,\beta)}(x+1,y;p) \tag{2.3}$$

Note that the right side of (2.3) appears as a closed form recurrence relation. From this one can see that a shift of the first variable by one is actually as if we take a quantity from the generalized beta function that equals to the integral of incomplete generalized beta function over the unit interval. Of course Theorem 1 could be rewritten so as to show the functional relation of the generalized beta function in its first variable if s is taken to be as an integer k to give

$$B^{(\alpha,\beta)}(x,y;p) - B^{(\alpha,\beta)}(x+k,y;p) = k\int_0^1 t^{k-1} B_t^{(\alpha,\beta)}(x,y;p)\,dt \tag{2.4}$$

In contrast with Theorem 1 where the weighted function is $t^{s-1}$, the weighted function in the following theorem is $(1-t)^{s-1}$.

**Theorem 2.** The following integral representation for generalized Beta function holds true:

$$\int_0^1 (1-t)^{s-1} B_t^{(\alpha,\beta)}(x,y;p)\,dt = \frac{1}{s} B^{(\alpha,\beta)}(x,y+s;p) \qquad (s \neq 0) \tag{2.5}$$

*Proof.* Again, we start with the integral form of $B^{(\alpha,\beta)}(x,y+s;p)$ and integrate it by parts by setting u = $(1-t)^s$ and the rest as dv.

**Corollary 2**. Taking s = 1 in Theorem 2 we get the following result:

$$\int_0^1 B_t^{(\alpha,\beta)}(x,y;p)\,dt = B^{(\alpha,\beta)}(x,y+1;p) \tag{2.6}$$

Note that the right side again appears in closed form in a simple and elegant result. In comparison to (2.3), the shift in the second variable by one is just the integral of incomplete generalized beta function over the unit interval. Indeed one can use (2.2) and (2.5) to get a more general form over the first and second variables of the generalized beta function.

**Theorem 3.** The following integral representations for generalized Beta function holds true:

$$B^{(\alpha,\beta)}(x+u,y+v;p) = B^{(\alpha,\beta)}(x,y+v;p) - u\int_0^1 t^{u-1} B_t^{(\alpha,\beta)}(x,y+v;p)\,dt \quad (u \neq 0) \tag{2.7}$$

$$B^{(\alpha,\beta)}(x+u,y+v;p) = v\int_0^1 (1-t)^{v-1} B_t^{(\alpha,\beta)}(x+u,y;p)\,dt \qquad (v \neq 0) \tag{2.8}$$

*Proof.* Replace y by y + v in (2.2) to give (2.7) and x by x + u in (2.5) to give (2.8).

**Corollary 3.** In the following corollary, the generalized beta function is written as the integral of an infinite sum of incomplete generalized beta function

$$B^{(\alpha,\beta)}(x,y;p) = \int_0^1 \sum_{n=0}^{\infty} B_t^{(\alpha,\beta)}(x+n,y;p)\,dt \tag{2.9}$$

*Proof.* In (2.8), put u = n and v = 1 to get

$$B^{(\alpha,\beta)}(x+n,y+1;p) = \int_0^1 B_t^{(\alpha,\beta)}(x+n,y;p)\,dt$$

Summing both sides and noting that the summation of the left side is simply the generalized beta function [5, p. 223 (5.72)], we get (2.9), as the order of summation and integration can be reversed as the integral is uniformly convergent.

**Theorem 4.** The following integral representation for generalized Beta function holds true:

$$\int_0^1 B_t^{(\alpha,\beta)}(x,y;p)\left[v(1-t)^{v-1}-ut\right]dt = B^{(\alpha,\beta)}(x+u,y;p)+B^{(\alpha,\beta)}(x,y+v;p)-B^{(\alpha,\beta)}(x,y;p)$$

$$(u,v\neq 0) \qquad (2.10).$$

## 3. APPLICATIONS TO THE GENERALIZED BETA FUNCTION

In this section we apply some results derived in pervious section. In particular, we apply Theorem 4 to obtain a relation between the complete version of the generalized beta function as the following theorem says.

**Theorem 5.** The following summation formula for generalized Beta function holds true:

$$B^{(\alpha,\beta)}(x,y;p)=\sum_{k=0}^{n}\binom{n}{k}B^{(\alpha,\beta)}(x+n-k,y+k;p) \qquad (3.1)$$

*Proof.* In (2.10) put u = v = 1 to obtain the well known result

$$B^{(\alpha,\beta)}(x+1,y;p)+B^{(\alpha,\beta)}(x,y+1;p)=B^{(\alpha,\beta)}(x,y;p) \qquad (3.2)$$

Now, use the shift iteratively to obtain (3.1).

Note that the result (3.1) applies also for p = 0 and hence it holds for the standard beta function and does not seem to appear in the literature. In the following theorem we obtain another remarkably elegant functional relation for the generalized beta function.

**Theorem 6.** The following functional relation for generalized Beta function holds true:

$$B^{(\alpha,\beta)}(x+n,y;p)-B^{(\alpha,\beta)}(x+1,y;p)=B^{(\alpha,\beta)}(x,y+n;p)-B^{(\alpha,\beta)}(x,y+1;p) \quad (n\geq 1) \qquad (3.3)$$

*Proof.* Take u = v = 2 in (2.11) to get

$$B^{(\alpha,\beta)}(x+2,y;p)+B^{(\alpha,\beta)}(x,y+2;p)=B^{(\alpha,\beta)}(x,y;p)+2\int_0^1 B_t^{(\alpha,\beta)}(x,y;p)(1-2t)\,dt \qquad (3.4)$$

Now, by using (2.2) with s = 2, (3.4) can be written as

$$B^{(\alpha,\beta)}(x, y+2; p) - B^{(\alpha,\beta)}(x+2, y; p) = B^{(\alpha,\beta)}(x, y; p) - 2B^{(\alpha,\beta)}(x+1, y; p) \tag{3.5}$$

By applying (3.2) to B(x, y; p) appearing on the right side of (3.5) and rearranging the terms, we get:

$$B^{(\alpha,\beta)}(x+2, y; p) - B^{(\alpha,\beta)}(x+1, y; p) = B^{(\alpha,\beta)}(x, y+2; p) - B^{(\alpha,\beta)}(x, y+1; p) \tag{3.6}$$

Hence our result is achieved if we apply this result (3.6) iteratively, i.e. for (x + 3) and (y + 3) in terms of (x + 2) and (y + 2), etc. This proves the theorem.
Again note that this result holds for the standard beta function and does not seem to appear in the literature.

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