ORIGINAL PAPER

HOW TO FIND THE INVERSE OF A MATRIX IN LORENTZ SPACE

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Abstract. *In the present paper, we define adjoint of a Lorentzian matrix. We give a theorem to find the inverse of a matrix using adjoint of it. Also a shortcut to find the inverse of a 2x2 matrix is presented. Finally, we show that the inverse of a matrix can be found by changing the identity matrix with L-identity matrix in the augmented matrix and using Gauss-Jordan elimination method. Furthermore, we introduce some new theorems related with adjoint and inverse of a matrix.*

Keywords: *Lorentz space, Lorentzian matrix, inverse of a matrix.*

1. INTRODUCTION

A real $n \times n$ matrix A is said to be Lorentzian if and only if associated linear transformation $A:\mathbb{R}^n \to \mathbb{R}^n$ defined by A(X) = AX is Lorentzian. The set of all Lorentzian $n \times n$ matrices together with matrix multiplication forms a group $O(1,n-1)$; called the Lorentz group of $n \times n$ matrices. The group $O(1,n-1)$ is naturally isomorphic to the group of Lorentz transformation of $\mathbb{R}^n$ (For more details see [3]). Gündoğan and Keçilioğlu [1] defined a new matrix multiplication by using Lorentzian inner product. With this multiplication they showed that $\mathbb{R}^n_n$ is an algebra with unit.

Inspired by the paper of Gündoğan and Keçilioğlu, we define adjoint of a Lorentzian matrix and some notions in Lorentz space. Also we express some methods to find the inverse of a square matrix and give some theorems.

2. PRELIMINARIES

Let us denote the set of all $m \times n$ matrices by $\mathbb{R}^m_n$. It is known that this set is a real vector space with matrix addition and multiplication (see [2]). Let L^n be the vector space $\mathbb{R}^n$ equipped with the Lorentzian inner product

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$$\langle x, y\rangle_L = -x_1y_1 + \sum_{i=2}^{n} x_i y_i \ , \ x = (x_1, x_{2,...,} x_n), \ y = (y_1, y_{2,...,} y_n).$$

Definition 1. A function $\phi : \mathbb{R}^n \to \mathbb{R}^n$ is a Lorentz transformation if and only if

$$\langle \phi(x), \phi(y)\rangle_L = \langle x, y\rangle_L, \ \forall x, y \in \mathbb{R}^n \ [3].$$

Definition 2. Let $A = [a_{ij}] \in \mathbb{R}_n^m$, $B = [b_{ik}] \in \mathbb{R}_p^n$. Matrix multiplication $"\cdot L"$ is defined by [1] as

$$A_{\cdot L}B = \left[-a_{i1}b_{1k} + \sum_{j=2}^{n} a_{ij}b_{jk} \right].$$

We denote $\mathbb{R}_n^m$ with L-multiplication by L_n^m.

Definition 3. An $n\times n$ L-identity matrix according to L-multiplication, denoted by I_n, is defined by [1] as

$$I_n = \begin{bmatrix} -1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & 1 \end{bmatrix}.$$

Definition 4. An $n\times n$ matrix A is called L-invertible if there exists an $n\times n$ matrix B such that $A_{\cdot L}B = B_{\cdot L}A = I_n$. Then B is called the L-inverse of A and is denoted by A^{-1} [1].

Definition 5. Transpose of a matrix $A = [a_{ij}] \in L_n^m$ is denoted by A^T and defined as $A^T = [a_{ji}] \in L_m^n$ [1].

Definition 6. L-determinant of a matrix $A = [a_{ij}] \in \mathbb{R}_n^n$ is denoted by det A and defined as

$$\det A = \sum_{\sigma\in S_n} s(\sigma)\, a_{\sigma(1)1} a_{\sigma(2)2} ... a_{\sigma(n)n} \ ,$$

where Sn is the set of all permutations of the set $\{1, 2, ..., n\}$ and $s(\sigma)$ is the sign of the permutation σ [1].

Theorem 1. $\forall A, B \in L_n^n$**,** $\det(A_{\cdot L}B) = -\det A \det B$ *[1].*

3. INVERSE OF A MATRIX

In this section, we present some methods to find the inverse of a matrix in Lorentz space. The first step is to define required notions for the study. Also, we give some new theorems related with adjoint and inverse of a matrix.

Definition 7. Let $A=\left[a_{ij}\right]\in L_n^m$ and α_{ij} be the cofactor of a_{ij}, i = 1, 2, …, m; j = 1, 2, …, n. L- adjoint of a matrix A is denoted by $adj_L A$ and

$$adj_L A=\begin{bmatrix} c_{11} & c_{12} & . & . & . & c_{1m} \\ c_{21} & c_{22} & . & . & . & c_{2m} \\ . & . & . & . & . & . \\ c_{n1} & c_{n2} & . & . & . & c_{nm} \end{bmatrix},$$

where $c_{ji}=\varepsilon\alpha_{ij}$ and $\varepsilon=\begin{cases} -1, 2\le j\le n, i=1, \\ -1, j=1, 2\le i\le m, \\ 1, otherwise. \end{cases}$

Theorem 2. Let $A\in L_n^m$. Then

$$A_{\cdot L}\left(adj_L A\right)=\left(adj_L A\right)_{\cdot L}A=\left(\det A\right)I_{n.}$$

Proof. Let $A=\left[a_{ij}\right]\in L_n^m$ and $adj_L A=\left[c_{ji}\right]\in L_n^m$. Then

$$A_{\cdot L}adj_L A=\begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nm} \end{bmatrix}\cdot L\begin{bmatrix} c_{11} & c_{12} & . & . & . & c_{1n} \\ c_{21} & c_{22} & . & . & . & c_{2n} \\ . & . & . & . & . & . \\ c_{n1} & c_{n2} & . & . & . & c_{nn} \end{bmatrix}$$

$$=\begin{bmatrix} a_{11} & a_{12} & . & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & . & a_{2n} \\ . & . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & . & a_{nn} \end{bmatrix}\cdot L\begin{bmatrix} \alpha_{11} & -\alpha_{21} & . & . & . & -\alpha_{n1} \\ -\alpha_{12} & \alpha_{22} & . & . & . & \alpha_{n2} \\ . & . & . & . & . & . \\ -\alpha_{1n} & \alpha_{2n} & . & . & . & \alpha_{nn} \end{bmatrix}$$

$$=\begin{bmatrix} -\det A & 0 & . & . & . & 0 \\ 0 & \det A & 0 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & \det A \end{bmatrix}$$

$$=diag\left(-\det A,\det A,...,\det A\right)$$

$$=\left(\det A\right)\begin{bmatrix} -1 & 0 & . & . & . & 0 \\ 0 & 1 & 0 & . & . & 0 \\ . & . & . & . & . & . \\ 0 & . & . & . & 0 & 1 \end{bmatrix}$$

$$=\left(\det A\right)I_n,$$

where $c_{ij}\left(i,j=1,2,...,n\right)$ as in definition 7. Similarly one can show that $adj_L A_{\cdot L}\ A=\left(\det A\right)I_n$.

Theorem 3. $A, B \in L_n^n$ be L-invertible matrices. Then

$$adj_L\left(A \cdot_L B\right) = -\left(adj_L B\right) \cdot_L \left(adj_L A\right).$$

Proof. Observe that

$$\begin{aligned} A \cdot_L B \cdot_L adj_L\left(A \cdot_L B\right) &= adj_L\left(A \cdot_L B\right) \cdot_L A \cdot_L B \\ &= \left(\det\left(A \cdot_L B\right)\right) I_n \\ &= -\left(\det A \det B\right) I_n, \\ A \cdot_L B \cdot_L \left(adjB \cdot_L adjA\right) &= A \cdot_L \left(B \cdot_L adj_L B\right) \cdot_L adj_L A \\ &= A \cdot_L \left(\left(\det B\right) I_n\right) \cdot_L adj_L A \\ &= \det B\left(A \cdot_L adj_L A\right) = \det B\left[\left(\det A\right) I_n\right] \\ &= \left(\det A \det B\right) I_n. \end{aligned}$$

So we have

$$A \cdot_L B \cdot_L adj\left(A \cdot_L B\right) = -A \cdot_L B \cdot_L \left(adj_L B \cdot_L adj_L A\right)$$

Since A and B are L-invertible matrices, if we L-multiply both sides of the last equation with A^{-1} and B^{-1} on the left, respectively, we obtain

$$adj_L\left(A \cdot_L B\right) = -\left(adj_L B\right) \cdot_L \left(adj_L A\right)$$

which completes the proof.

Theorem 4. Let $A \in L_n^n$ be a L-invertible matrix. Then

$$\det\left(adj_L A\right) = \left(\det A\right)^{n-1}$$

Proof. By theorem 2 we have

$$\det\left(A \cdot_L adj_L A\right) = \det\left(\left(\det A\right) I_n\right) = \left(\det A\right)^n \det I_n = -\left(\det A\right)^n.$$

On the other hand by theorem 1

$$\det\left(A \cdot_L adj_L A\right) = -\det A \det\left(adj_L A\right).$$

Combining these two equations we obtain $\det\left(adj_L A\right) = \left(\det A\right)^{n-1}$ and it completes the proof.

Now we are ready to present following theorem to find the inverse of a matrix using adjoint of it.

Theorem 5. Let $A \in L_n^n$ be a L-invertible matrix. Then

$$A^{-1} = \frac{1}{\det A} adj_L A.$$

Proof. By theorem 2 we have $A \cdot_L adj_L A = \left(\det A\right) I_n$. Multiplying both sides with $\frac{1}{\det A} A^{-1}$ on the left, we obtain the required equality.

Example 1. The following steps result in A^{-1} for $A=\begin{bmatrix}1 & -1 & 2\\ 2 & 1 & -3\\ 4 & 1 & 1\end{bmatrix}\in L_3^3$. The L-adjoint matrix for A is $adj_L A=\begin{bmatrix}4 & -3 & -1\\ 14 & -7 & 7\\ 2 & -5 & 3\end{bmatrix}$. Since det A = 14, we get

$A^{-1}=\begin{bmatrix}2/7 & -3/14 & -1/14\\ 1 & -1/2 & 1/2\\ 1/7 & -5/14 & 3/14\end{bmatrix}$. It is easy to see that $A\cdot_L A^1 = A^{-1}\cdot_L A = I_3$.

Example 2. (Inverse of Lorentz transformation in Lorentz space)
The Lorentz transformation

$$x' = \gamma\left(x - vt\right)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\left(t - vx/c^2\right)$$

or

$$\begin{bmatrix}x'\\ y'\\ z'\\ t'\end{bmatrix}=\begin{bmatrix}-\gamma & 0 & 0 & -\gamma v\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \gamma v/c^2 & 0 & 0 & \gamma\end{bmatrix}\cdot_L\begin{bmatrix}x\\ y\\ z\\ t\end{bmatrix}$$

is a linear coordinate transformation in the space (x; y; z; t) and can be represented by a symmetric 4x4 matrix $\hat{L}$ with components given by

$$\hat{L}=\begin{bmatrix}-\gamma & 0 & 0 & -\gamma \upsilon\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \gamma \upsilon/c^2 & 0 & 0 & \gamma\end{bmatrix}$$

for scalars υ, c, $\gamma=\dfrac{1}{\sqrt{1-\dfrac{\upsilon^2}{c^2}}}$. Since the $\det\left(\hat{L}\right)=1$, we find the *L*-inverse of the matrix $\hat{L}$

$$\hat{L}^{-1}=adj_L\hat{L}=\begin{bmatrix}-\gamma & 0 & 0 & -\gamma \upsilon\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ \gamma \upsilon/c^2 & 0 & 0 & \gamma\end{bmatrix}.$$

In fact

$$\hat{L}\cdot_L\hat{L}^{-1}=\hat{L}^{-1}\cdot_L\hat{L}=I_n.$$

3.1. Shortcut for 2x2 matrice: For $A=\begin{bmatrix} a & b \\ c & d \end{bmatrix}\in L_2^2$ the L-inverse may be found using the formula:

$$A^{-1}=\frac{1}{\det A}\begin{bmatrix} d & b \\ c & a \end{bmatrix}$$

Example 3. $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1}=-\frac{1}{2}\begin{bmatrix} 4 & 2 \\ 3 & 1 \end{bmatrix}=\begin{bmatrix} -2 & -1 \\ 3 & -1/2 \end{bmatrix}$.

3.2. Augumented matrix method: Last method to find the L-inverse of a matrix is the augmented matrix method. One can use Gauss-Jordan elimination to transform $\left[A \,\vdots\, I\right]$ to $\left[I \,\vdots\, A^{-1}\right]$.

Example 4. $\left[A \,\vdots\, I\right]=\left[\begin{matrix} 1 & 2 \\ 3 & 4 \end{matrix}\,\vdots\,\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}\right]\xrightarrow{-3R_1+R_2}\left[\begin{matrix} 1 & 2 \\ 0 & -2 \end{matrix}\,\vdots\,\begin{matrix} -1 & 0 \\ 3 & 1 \end{matrix}\right]$

$$\xrightarrow{-R_2/2}\left[\begin{matrix} 1 & 2 \\ 0 & 1 \end{matrix}\,\vdots\,\begin{matrix} -1 & 0 \\ -3/2 & -1/2 \end{matrix}\right]\xrightarrow{-2R_2+R_1}\left[\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}\,\vdots\,\begin{matrix} 2 & 1 \\ -3/2 & -1/2 \end{matrix}\right]$$

$$\xrightarrow{-R_1}\left[\begin{matrix} -1 & 0 \\ 0 & 1 \end{matrix}\,\vdots\,\begin{matrix} -2 & -1 \\ -3/2 & -1/2 \end{matrix}\right]=\left[I \,\vdots\, A^{-1}\right]$$

REFERENCES

[1] Gündogan, H., Keçilioglu, O., *Glasnik Matematicki*, **41**(61), 329, 2006.
[2] Lang, S., *Linear Algebra*, Addison-Wesley, 1971.
[3] Ratcliffe, J.G., *Foundations of hyperbolic manifolds*, 2nd Edition, Springer, 2006.