ORIGINAL PAPER

SURFACE FAMILY WITH A COMMON NATURAL LINE OF CURVATURE LIFT

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Abstract. *We construct a surface family possessing a natural lift of a given curve as a line of curvature. We obtain necessary and sufficient condition for the given curve such that its natural lift is a line of curvature on any member of the surface family. Finally, we present some illustrative examples.*
Keywords: *suface curve, curvature lift, differential geometry.*

1. INTRODUCTION AND PRELIMINARIES

We encounter curves and surfaces in every differential geometry book. Regardless of the representation of the surface, most existing work deal with the classification of surface curves. However, the more relevant problem is constructing surfaces upon a given curve possessing it as a special curve rather than finding and classifying surface curves.

The first paper related with this type of problem proposed by Wang et.al. [1]. They constructed surfaces passing through a given curve as common geodesic. In 2011, a similar paper published by Li et.al. [2] who handled the problem of finding surfaces with a common line of curvature. Bayram et.al. [3] tackled the problem of constructing surfaces passing through a given asymptotic curve. In [4] authors studied surfaces with a null asymptotic curve. Recently, Bayram et.al. [5] found constraints for the natural lift of a given curve to be an asymptotic curve on the surface family.

Inspired with the above papers, we search for a surface family possessing the natural lift of a given curve as a common line of curvature. We obtain the necessary and sufficient condition for the resulting surface to have the natural lift of a given curve as a common line of curvature.

We start with giving some background about the subject.

A parametric curve $\alpha(s)$, $L_1 \le s \le L_2$, is a curve on a surface $P(s,t)$ in R^3 that has a constant s or t-parameter value. In this paper, α' denotes the derivative of α with respect to arc length parameter s and we assume that α is a regular curve with $\alpha''(s) \ne 0$, $L_1 \le s \le L_2$. For every point of $\alpha(s)$, the set $\{T(s), N(s), B(s)\}$ is called the Frenet frame along $\alpha(s)$, where $T(s) = \alpha'(s)$, $N(s) = \frac{\alpha''}{\|\alpha''\|}$ and $B(s) = T(s) \times N(s)$ are the unit tangent, principal normal, and binormal vectors of the curve at the point $\alpha(s)$, respectively. Derivative formulas of the Frenet frame is governed by the relations

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$$\frac{d}{ds}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix} = \begin{pmatrix} 0 & \kappa(s) & 0 \\ -\kappa(s) & 0 & \tau(s) \\ 0 & -\tau(s) & 0 \end{pmatrix}\begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{1}$$

where $\kappa(s) = \|\alpha''(s)\|$ and $\tau(s) = -\langle B'(s), N(s)\rangle$ are called the curvature and torsion of the curve $\alpha(s)$, respectively [6].

Let P be a surface in R^3 and let $\alpha : I \to P$ be a parametrized curve. α is called an integral curve of X if

$$\frac{d}{ds}(\alpha(s)) = X(\alpha(s)) \text{ (for all t∈I)},$$

where X is a smooth tangent vector field on P. We have

$$TP = \bigcup_{p \in P} T_p P = \chi(P),$$

where $T_p P$ is the tangent space of P at p and $\chi(P)$ is the space of vector fields on P.

For any parametrized curve $\alpha : I \to P$, $\overline{\alpha} : I \to TP$ given by

$$\overline{\alpha}(s) = (\alpha(s), \alpha'(s)) = \alpha'(s)|_{\alpha(s)}$$

is called the natural lift of α on TP [7]. Thus, we can write $\frac{d\overline{\alpha}}{ds} = \frac{d}{ds}\left(\alpha'(s)|_{\alpha(s)}\right)$.

A regular curve α on P is said to be a line of curvature of P if for all $p \in \alpha$ the tangent line of α is a principal direction at p. According to this definition, the differential equation of the line of curvature on P is $S(T) = \lambda T,\ T \neq 0,$ where S is the shape operator of P.

If a rigid body moves along a unit speed curve $\alpha(s)$, then the motion of the body consists of translation along α and rotation about α. The rotation is determined by an angular velocity vector ω which satisfies $T' = \omega \times T,\ N' = \omega \times N$ and $B' = \omega \times B$. The vector ω is called the *Darboux vector*. In terms of Frenet vectors T, N and B, Darboux vector is given by $\omega = \tau T + \kappa B$ [8]. Also, we have $\kappa = \|\omega\|\cos\theta,\ \tau = \|\omega\|\sin\theta,$ where θ is the angle between the Darboux vector of α and binormal vector $B(s)$. Observe that $\theta = \arctan\frac{\tau}{\kappa}$ (Fig. 1).

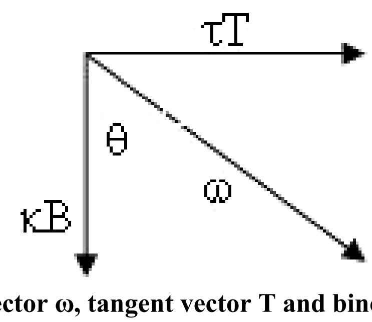

Figure 1. Darboux vector ω, tangent vector T and binormal vector B of α.

Let $\alpha(s),\ L_1 \le s \le L_2$, be an arc length curve and $\overline{\alpha}(s),\ L_1 \le s \le L_2$, be the natural lift of α. Then we have

$$\begin{pmatrix} \bar{T}(s) \\ \bar{N}(s) \\ \bar{B}(s) \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ -\cos\theta & 0 & \sin\theta \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} T(s) \\ N(s) \\ B(s) \end{pmatrix}, \tag{2}$$

where $\{T(s), N(s), B(s)\}$ and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ are Frenet frames of the curves α and $\bar{\alpha}$, respectively, and θ is the angle between the Darboux vector ω and binormal vector B of α.

2. SURFACE FAMILY WITH A COMMON NATURAL LINE OF CURVATURE LIFT

Suppose we are given a 3-dimensional parametric curve $\alpha(s)$, $L_1 \le s \le L_2$, in which s is the arc length and $\|\alpha''(s)\| \ne 0$, $L_1 \le s \le L_2$. Let $\bar{\alpha}(s)$, $L_1 \le s \le L_2$, be the natural lift of the given curve $\alpha(s)$.

Surface family that interpolates $\bar{\alpha}(s)$ as a common curve is given in the parametric form as

$$P(s,t) = \bar{\alpha}(s) + u(s,t)\bar{T}(s) + v(s,t)\bar{N}(s) + w(s,t)\bar{B}(s), \tag{3}$$

where $u(s,t)$, $v(s,t)$ and $w(s,t)$ are C^1 functions and are called *marching-scale functions* and $\{\bar{T}(s), \bar{N}(s), \bar{B}(s)\}$ is the Frenet frame of the curve $\bar{\alpha}$. Using Eqn. (2) we can express Eqn. (3) in terms of Frenet frame $\{T(s), N(s), B(s)\}$ of the curve α as

$$\begin{aligned} P(s,t) = &\big(1 - v(s,t)\cos\theta + w(s,t)\sin\theta\big)T(s) \\ &+ u(s,t)N(s) + \big(v(s,t)\sin\theta + w(s,t)\cos\theta\big)B(s), \end{aligned} \tag{4}$$

where $L_1 \le s \le L_2$, $T_1 \le t \le T_2$.

Remark 1. Observe that choosing different marching-scale functions yields different surfaces possessing $\bar{\alpha}(s)$ as a common curve.

Our goal is to find the necessary and sufficient conditions for which the curve $\bar{\alpha}(s)$ is isoparametric and line of curvature on the surface $P(s,t)$. Firstly, as $\bar{\alpha}(s)$ is an isoparametric curve on the surface $P(s,t)$, there exists a parameter $t_0 \in [T_1, T_2]$ such that

$$u(s,t_0) = v(s,t_0) = w(s,t_0) \equiv 0,\ L_1 \le s \le L_2,\ T_1 \le t_0 \le T_2. \tag{5}$$

Secondly, we search for the line of curvature condition. Before, we give the following theorem:

Theorem 2. A necessary and sufficient condition that a surface curve be a line of curvature is that the surface normals along the curve form a developable surface [9].

Let $n_1(s)=(\cos\phi)\bar{N}(s)+(\sin\phi)\bar{B}(s)$ be a vector orthogonal to the curve $\bar{\alpha}(s)$, where $\phi=\phi(s)$ is the angle between $\bar{N}$ and n_1. Using Eqn. (2) we have $n_1(s)=(\cos(\theta+\phi))T(s)+(\sin(\theta+\phi))B(s)$. The curve $\bar{\alpha}(s)$ is a line of curvature on the surface $P(s,t)$ if and only if n_1 is parallel to the normal vector $n(s,t)$ of the surface $P(s,t)$ and the ruled surface

$$Q(s,t)=\bar{\alpha}(s)+tn_1(s),\ L_1\le t\le L_2 \tag{6}$$

is developable.

We first derive the condition for $n_1(s)$ to be parallel to the normal vector $n(s,t)$ of the surface $P(s,t)$:

The normal vector can be expressed as

$$n(s,t)=\frac{\partial P(s,t)}{\partial s}\times\frac{\partial P(s,t)}{\partial t}.$$

Along the curve $\bar{\alpha}$ the normal vector reduces to

$$n(s,t_0)=\kappa\left[-\frac{\partial w}{\partial t}(s,t_0)\bar{N}(s)+\frac{\partial v}{\partial t}(s,t_0)\bar{B}(s)\right],$$

where κ is the curvature of the curve α. This follows that $n_1(s)//n(s,t_0)$, $L_1\le s\le L_2$, if and only if there exits a function $\sigma(s)\ne 0$ such that

$$\frac{\partial w}{\partial t}(s,t_0)=-\sigma(s)\cos\phi(s),\ \frac{\partial v}{\partial t}(s,t_0)=\sigma(s)\sin\phi(s). \tag{7}$$

Secondly, surface (6) is developable if and only if $\det(\bar{\alpha}',n_1,n_1')=0$ [6]. After simple computation, we get

$$\det(\bar{\alpha}',n_1,n_1')=0\Leftrightarrow\theta+\phi=\text{constant}. \tag{8}$$

Combining (5), (7) and (8), we have the following theorem.

Theorem 3. The natural lift curve $\bar{\alpha}(s)$ is a line of curvature on the surface (3) if and only if the followings are satisfied:

$$\begin{cases} u(s,t_0)=v(s,t_0)=w(s,t_0)\equiv 0, \\ \theta(s)+\phi(s)=\text{constant}, \\ \frac{\partial w}{\partial t}(s,t_0)=-\sigma(s)\cos\phi(s),\ \frac{\partial v}{\partial t}(s,t_0)=\sigma(s)\sin\phi(s), \end{cases}$$

where $L_1\le s\le L_2$, $\exists t_0\in[T_1,T_2]$, $\sigma(s)\ne 0$.

2.1. EXAMPLES

Example 1. Let $\alpha(s)=(\cos s,\sin s,0)$ be a unit speed curve. Then, it is easy to show that

$$T(s)=(-\sin s,\cos s,0),$$
$$N(s)=(-\cos s,-\sin s,0),$$
$$B(s)=(0,0,1),$$
$$\kappa=1,\ \tau=0,\ \theta=0.$$

We have

$$\overline{\alpha}(s)=(-\sin s,\cos s,0)$$

as the natural lift of α with Frenet vectors

$$\overline{T}(s)=(-\cos s,-\sin s,0),$$
$$\overline{N}(s)=(\sin s,-\cos s,0),$$
$$\overline{B}(s)=(0,0,1).$$

If we choose $u(s,t)=t,\ v(s,t)=\frac{\sqrt{2}}{2}t,\ w(s,t)=-\frac{\sqrt{2}}{2}t,\ \sigma(s)\equiv 1,\ \phi(s)\equiv\frac{\pi}{4}$ and $t_0=0,$ then Theorem 3 is satisfied and we get the surface

$$P_1(s,t)=\overline{\alpha}(s)+u(s,t)\overline{T}(s)+v(s,t)\overline{N}(s)+w(s,t)\overline{B}(s)$$
$$=\left(\left(\frac{\sqrt{2}}{2}t-1\right)\sin s-t\cos s,\right.$$
$$\left.\left(1-\frac{\sqrt{2}}{2}t\right)\cos s-t\sin s,-\frac{\sqrt{2}}{2}t\right),$$

$-5\le s\le 5,\ -1\le t\le 0,5,$ possessing $\overline{\alpha}$ as a line of curvature (Fig. 2).

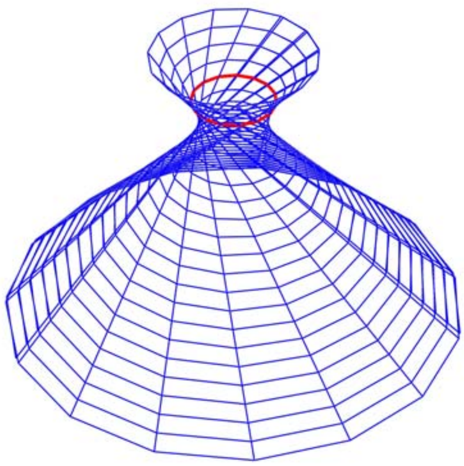

Figure 2. $P_1(s,t)$ **as a member of the surface family and its common natural line of curvature lift** $\overline{\alpha}$**.**

For the same curve, if we take $u(s,t)=e^t-1,\ v(s,t)=\frac{st}{2},\ w(s,t)=-\frac{\sqrt{3}}{2}st,\ \sigma(s)=s,\ \phi(s)\equiv\frac{\pi}{6}$ and $t_0=0,$ then Theorem 3 is satisfied and we obtain the surface

$$P_2(s,t)=\left(\left(\frac{st}{2}-1\right)\sin s-\left(e^t-1\right)\cos s,\right.$$
$$\left.\left(1-\frac{st}{2}\right)\cos s-\left(e^t-1\right)\sin s,-\frac{\sqrt{3}}{2}st\right),$$

$0<s\le 2,\ 0\le t\le 2,$ possessing $\bar{\alpha}$ as a line of curvature $(\text{Fig. }3)$.

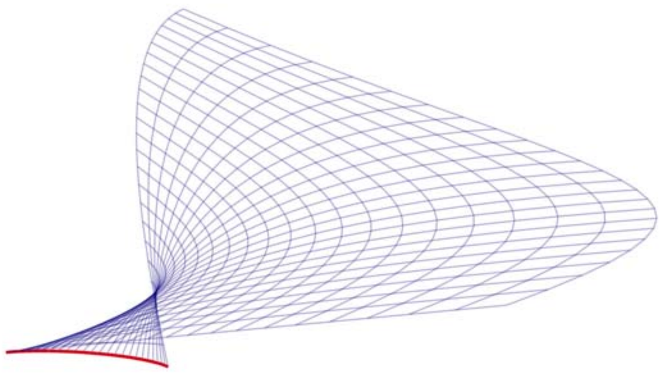

Figure 3. $P_2(s,t)$ **as a member of the surface family and its common natural line of curvature lift** $\bar{\alpha}$ **.**

Example 2. Consider the arc length helix $\alpha(s)=\left(\cos\left(\frac{\sqrt{2}}{2}s\right),\sin\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}s\right)$. One can easily compute

$$T(s)=\left(-\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\right),$$
$$N(s)=\left(-\cos\left(\frac{\sqrt{2}}{2}s\right),-\sin\left(\frac{\sqrt{2}}{2}s\right),0\right),$$
$$B(s)=\left(\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right),-\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\right),$$
$$\kappa=\tau=\frac{1}{2},\ \theta=\frac{\pi}{4}.$$

The natural lift of α is

$$\bar{\alpha}(s)=\left(-\frac{\sqrt{2}}{2}\sin\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\cos\left(\frac{\sqrt{2}}{2}s\right),\frac{\sqrt{2}}{2}\right)$$

with Frenet vectors

$$\bar{T}(s)=\left(-\cos\left(\frac{\sqrt{2}}{2}s\right),-\sin\left(\frac{\sqrt{2}}{2}s\right),0\right),$$
$$\bar{N}(s)=\left(\sin\left(\frac{\sqrt{2}}{2}s\right),-\cos\left(\frac{\sqrt{2}}{2}s\right),0\right),$$
$$\bar{B}(s)=(0,0,1).$$

If we choose $u(s,t)=st^2,\ v(s,t)=\frac{\sqrt{2}}{2}st,\ w(s,t)=-\frac{\sqrt{2}}{2}st,\ \sigma(s)=s,\ \phi(s)\equiv\frac{\pi}{4}$ and $t_0=0,$ then Theorem 3 is satisfied and we obtain the surface

$$P_3(s,t) = \left(\left(\frac{\sqrt{2}}{2} st - 1 \right) \frac{\sqrt{2}}{2} \sin\left(\frac{\sqrt{2}}{2} s \right) - st^2 \sin\left(\frac{\sqrt{2}}{2} s \right), \right.$$

$$\left. \left(1 - \frac{\sqrt{2}}{2} st \right) \frac{\sqrt{2}}{2} \cos\left(\frac{\sqrt{2}}{2} s \right) - st^2 \cos\left(\frac{\sqrt{2}}{2} s \right), \frac{\sqrt{2}}{2}(1 - st) \right),$$

$0 < s \leq 2,\ -1 \leq t \leq 1,$ possessing $\bar{\alpha}$ as a line of curvature $(\text{Fig. } 4)$.

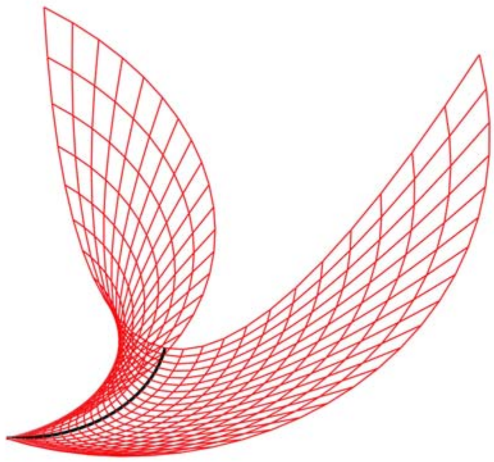

Figure 4. $P_3(s,t)$ **as a member of the surface family and its common natural line of curvature lift** $\bar{\alpha}$.

For the same curve, if we choose $u(s,t) \equiv 0,\ v(s,t) = e^s t^2,\ w(s,t) = -e^s t,\ \sigma(s) = e^s,\ \phi(s) \equiv 0$ and $t_0 = 0,$ then Theorem 3 is satisfied and we obtain the surface

$$P_4(s,t) = \left(\left(e^s t^2 - \frac{\sqrt{2}}{2} \right) \sin\left(\frac{\sqrt{2}}{2} s \right), \right.$$

$$\left. \left(\frac{\sqrt{2}}{2} - e^s t^2 \right) \cos\left(\frac{\sqrt{2}}{2} s \right), \frac{\sqrt{2}}{2} - e^s t \right),$$

$-1 \leq s,t \leq 1,$ possessing $\bar{\alpha}$ as a line of curvature $(\text{Fig. } 5)$.

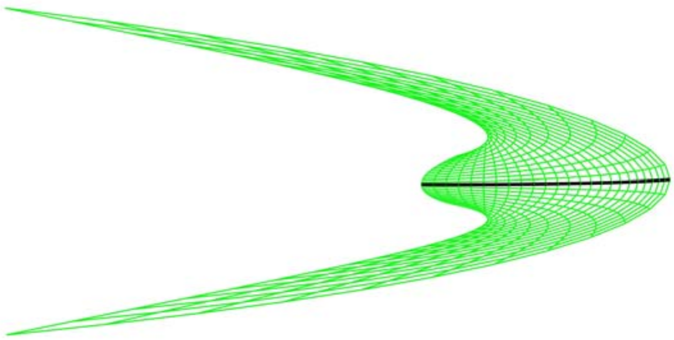

Fig. 5 : $P_4(s,t)$ **as a member of the surface family and its common natural line of curvature lift** $\bar{\alpha}$ **.**

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REFERENCES

[1] Wang, G.J., Tang, K., Tai, C.L., *Comput. Aided Des.*, **36**(5), 447, 2004.
[2] Li, C.Y., Wang, R.H., Zhu, C.G., *Comput. Aided Des.*, **43**(9), 1110, 2011.
[3] Bayram, E., Güler, F., Kasap, E., *Comput. Aided Des.*, **44**, 637, 2012.
[4] Atalay, G.Ş., Kasap, E., *Appl. Math. Comput.*, **260**, 135, 2015.
[5] Bayram, E., Ergün, E., Kasap, E., *J. Sci. and Arts*, **2**(31), 117, 2015.
[6] do Carmo, M.P., *Differential geometry of curves and surfaces,* Englewood Cliffs (New Jersey): Prentice Hall Inc., 1976.
[7] Thorpe, J.A., *Elementary topics in differential geometry,* Springer-Verlag, New York, Heidelberg-Berlin, 1979.
[8] Oprea, J., *Differential geometry and its applications,* Pearson Education Inc., USA, 2007.
[9] Willmore, T.J., *An introduction to differential geometry*, Oxford University Press, 1959.