ORIGINAL PAPER

RESEARCH ARTICLE ASSESSING HIGH SCHOOL STUDENTS' MATHEMATICS COMPETENCY: USING VISUAL IMAGE OF CONVEX FUNCTION GRAPH, CONCAVE FUNCTION TO PROVE THE INEQUALITY

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Abstract. *The main purpose of the article is to introduce an active teaching method, based on the nature of a familiar mathematical concept. Specifically, this paper presents the application of the visual image of convex function and concave function graphs concave function on demonstrating some inequality in the primary curriculum. Also, some results and comments concerning the proving method of such as inequalities are stated.*

Keywords: *Inequality, Convex function, Concave function.*

1. INTRODUCTION

With innovative approaches to teaching high school, student-centered and create excitement in learning. Students actively dominate knowledge. Therefore, teaching the student grasp the essence of a mathematical concept is very important.

2. RESULTS AND DISCUSSION

Level 1. The teachers assess students' competency of recalling knowledge

Firstly, we recall some basis concepts, that is

Definition 1. *Let* $y = f(x)$ *be a continous function on* $[a,b]$ *that has graph (C). Suppose that* $A(a; f(a)), B(b; f(b))$ *are two points lie on the* (C). *Then*

a) The graph (C) is called convex on (a,b) if the tangent at any point on the arc AB is always located on the upper graph (C);

b) The graph (C) is called concave on (a, b) if the tangent at any point on the arc AB is always located on the upper graph (C).

Theorem 1. *Suppose that* $f \in C^2(a,b)$. *Then, we have*

* If $f''(x) > 0$ with all $x \in (a,b)$, then its graph is concave on (a,b);
* If $f''(x) < 0$ with all $x \in (a,b)$, then its graph is convex on (a,b).

Theorem 2. *Let* $f = f(x)$ *be a continous function and having secondary derivative on* $[a,b]$. *Then,*

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a) If $f''(x) \geq 0, \forall x \in [a,b]$, then $f(x) \geq f'(x_0)(x-x_0)+f(x_0), \quad \forall x_0 \in [a,b]$;
b) If $f''(x) \leq 0, \forall x \in [a,b]$, then $f(x) \leq f'(x_0)(x-x_0)+f(x_0), \quad \forall x_0 \in [a,b]$.

Equality of the two above inequality occurs if and only if $x = x_0$.

Theorem 3. *Let $f = f(x)$ be a continous function and having secondary derivative on $[a,b]$. Then,*

a) If $f''(x) \geq 0, \forall x \in [a,b]$, then $f(x) \geq \dfrac{f(a)-f(b)}{a-b}(x-a)+f(a), \quad \forall x_0 \in [a,b]$;
b) If $f''(x) \leq 0, \forall x \in [a,b]$, then $f(x) \leq \dfrac{f(a)-f(b)}{a-b}(x-a)+f(a), \quad \forall x_0 \in [a,b]$.

Equality of the two above inequality occurs if and only if $x = a$ or $x = b$.

Example 1. Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{\sqrt{a^2+1}}+\frac{b}{\sqrt{b^2+1}}+\frac{c}{\sqrt{c^2+1}} \leq \frac{3}{\sqrt{10}}.$$

Proof. By considering the function $f(x)=\dfrac{x}{\sqrt{x^2+1}}$, with $x \in (0,1)$. Then we have

$$f'(x)=\frac{1}{(\sqrt{x^2+1})^3} \Rightarrow f''(x)=-\frac{3x}{(\sqrt{x^2+1})^5}<0, \forall x \in (0,1).$$

Applying the theorem 0.2, we obtain:

$$f(a) \leq f'(\frac{1}{3})(a-\frac{1}{3})+f(\frac{1}{3})$$

$$f(b) \leq f'(\frac{1}{3})(b-\frac{1}{3})+f(\frac{1}{3})$$

$$f(c) \leq f'(\frac{1}{3})(c-\frac{1}{3})+f(\frac{1}{3})$$

Combining (1), (2), and (3) to get $f(a)+f(b)+f(c) \leq f'(\frac{1}{3})(a+b+c-1)+3f(\frac{1}{3})=\dfrac{3}{\sqrt{10}}$.

Level 2. The teachers assess students' competency at higher level as they give hypothesis and prove them.

From the examples 1 above, students may find that:

Remark 1. Signs help us realize the above method is to prove the inequality of the form

$$f(x_1)+f(x_2)+f(x_3)+....+f(x_n) \geq k; \quad f(x_1)+f(x_2)+f(x_3)+....+f(x_n) \leq k$$

where, $a_i \in \mathbb{R}, \forall i=1,2,..n$. In some cases, inequality has not spread out, we have to make a few new transformations, put it in basic form. That is

* If the inequality of the form $f(x_1)f(x_2)f(x_3)....f(x_n) \geq k$, then we take loganepe two sides.
* If inequality must demonstrate is homogenised, then we can normalize. Depending on the problem that we choose to standardize accordingly.

Example 2. Let a,b,c be positive real numbers such that $a+b+c=3$. Prove that

$$P=\left(a+\sqrt{1+a^2}\right)^b\left(b+\sqrt{1+b^2}\right)^c\left(c+\sqrt{1+c^2}\right)^a \leq \left(1+\sqrt{2}\right)^3.$$

Proof. We have $\ln P = b\ln\left(a+\sqrt{1+a^2}\right)+c\ln\left(b+\sqrt{1+b^2}\right)+a\ln\left(c+\sqrt{1+c^2}\right)$. Considering function: $f(x)=\ln\left(x+\sqrt{1+x^2}\right)$, $0<x<1$.

We have $f'(x)=\dfrac{1}{\sqrt{x^2+1}} \Rightarrow f''(x)=\dfrac{-x}{\sqrt{(1+x^2)}^3}<0, \forall x\in(0,1)$, and so $f(a)\le f'(1)(a-1)+f(1)=f'(1)a+f(1)-f'(1)$ or $bf(a)\le f'(1)ab+[f(1)-f'(1)]b$. Similarly, we also have $cf(b)\le f'(1)cb+[f(1)-f'(1)]c$; $af(c)\le f'(1)ac+[f(1)-f'(1)]a$. Combining (4), (5), (6), we obtain $\ln P\le f'(1)\left(ab+bc+ca-\left(a+b+c\right)\right)+f(1)\left(a+b+c\right)$, with $ab+bc+ca\le\dfrac{(a+b+c)^2}{3}=3$, and therefore $\ln P\le 3\ln(1+\sqrt{2})\Rightarrow P\le(1+\sqrt{2})^3$.

This completes the proof.

Example 3. Let $x,y,z>0$ are positive real numbers. Prove that

$$\frac{1+\sqrt{3}}{3\sqrt{3}}\left(x^2+y^2+x^2\right)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\ge x+y+z+\sqrt{x^2+y^2+z^2}.$$

Proof. Because inequality has to be homogeneous, so we just need to prove it true for all positive real numbers a,b,c satisfying $x^2+y^2+z^2=1$, Meanwhile, inequality should prove to become $f(x)+f(y)+f(z)\ge 1$, where, $f(x)=\dfrac{1+\sqrt{3}}{3\sqrt{3}}\dfrac{1}{x}-x$, with $0<x<1$. Noticing that $f''(x)>0, \forall x\in(0,1)$, therefore applying the theorem 1, we obtain $f(x)+f(y)+f(z)\ge f'(\frac{1}{\sqrt{3}})\left(x+y+z-\sqrt{3}\right)+3f\left(\dfrac{1}{\sqrt{3}}\right)$.By using $f'(\frac{1}{\sqrt{3}})<0$ and $x+y+z\le\sqrt{3\left(x^2+y^2+z^2\right)}=\sqrt{3}$, we have $f(x)+f(y)+f(z)\ge 3f\left(\dfrac{1}{\sqrt{3}}\right)=1$.

Level 3. Teachers teach students the ability to generalize the problem by example

Theorem 4. *Let function $y=f(x)$,which has secondary derivative on $[a,b]$ and n real numbers $a_1,a_2,...,a_n\in[a,b]$ satisfying $\sum_{i=1}^{n}a_i=k, na\le k\le nb$. Then we have*

a) If $f''(x)>0$ with $\forall x\in[a,b]$ then $\sum_{i=1}^{n}f(a_i)\ge nf\left(\dfrac{k}{n}\right)$;

b) If $f''(x)<0$ with $\forall x\in[a,b]$ then $\sum_{i=1}^{n}f(a_i)\le\dfrac{1}{n}f\left(\dfrac{k}{n}\right)$.

Example 4. Let ΔABC , with $A\ge\dfrac{2\pi}{3}>B\ge C$. Prove that $\tan\dfrac{A}{2}+\tan\dfrac{B}{2}+\tan\dfrac{C}{2}\ge 4-\sqrt{3}$

Proof. From assumption, we have $C\le\dfrac{\pi}{6}$. Considering function $f(x)=\tan x, x\in\left(0,\dfrac{\pi}{3}\right)$ having $f''(x)>0$ with $\forall x\in\left(0,\dfrac{\pi}{3}\right)$. Then we have

$$f(\frac{A}{2}) \geq f'\left(\frac{\pi}{3}\right)\left(\frac{A}{2}-\frac{\pi}{3}\right)+f\left(\frac{\pi}{3}\right)$$
$$f(\frac{B}{2}) \geq f'\left(\frac{\pi}{12}\right)\left(\frac{B}{2}-\frac{\pi}{12}\right)+f\left(\frac{\pi}{12}\right)$$
$$f(\frac{C}{2}) \geq f'\left(\frac{\pi}{12}\right)\left(\frac{C}{2}-\frac{\pi}{12}\right)+f\left(\frac{\pi}{12}\right)$$

This implies

$$f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{B}{2}\right) \geq \left[f'\frac{\pi}{3}-f'\frac{\pi}{12}\right]\left(\frac{A}{2}-\frac{2\pi}{3}\right)+f'\left(\frac{\pi}{12}\right)\left(\frac{A+B+C}{2}-\frac{\pi}{2}\right)$$
$$+f\left(\frac{\pi}{3}\right)+2f\left(\frac{\pi}{12}\right)$$

Note that $f'\left(\frac{\pi}{3}\right)-f'\left(\frac{\pi}{12}\right)>0; \frac{A}{2}-\frac{\pi}{3}\geq 0$ and $\frac{A+B+C}{2}=\frac{\pi}{2}$, and so $f\left(\frac{A}{2}\right)+f\left(\frac{B}{2}\right)+f\left(\frac{C}{2}\right)\geq f\left(\frac{\pi}{3}\right)+2f\left(\frac{\pi}{12}\right)=4-\sqrt{3}$. Equality occurs if and only if $A=\frac{2\pi}{3}; B=C=\frac{\pi}{6}$ and the permutations.

Level 4. Teachers teach students the capacity to analyze problems

In some cases, may be exist $x_0 \in [a,b]$ such that the graph of function $f(x)$ is convex on (a, x_0) and concave on (x_0, b), but we can still use the inequality

$$f(x) \geq f^{'}(x_0)(x-x_0)+f(x_0), x_0 \in (a,b).$$

Example 5. Let $a,b,c \in \mathbb{R}$ and $a+b+c=6$. Prove that

$$a^4+b^4+c^4 \geq 2(a^3+b^3+c^3).$$

Proof. We have $a^4+b^4+c^4 \geq 2(a^3+b^3+c^3) \Leftrightarrow \left(a^4-2a^3\right)+\left(b^4-2b^3\right)+\left(c^4-2c^3\right).$

Considering the function $f(x)=x^4-2x^3$, having $f''(x)=12x^2-2x^3$. Then, we can not use directly tangential inequality, we can still assess through its tangent at the point $M(x_0, f(x_0)), x_0=2$, that is $y=8x-16$. We have $f(x)-(8x-16)=x^4-2x^3-8x+16=(x-2)^2(x^2-2x+4), \forall x \in \mathbb{R}$. Therefore, we obtain $f(a)+f(b)+f(c) \geq 8(a+b+c)-48=0$. This means $a^4+b^4+c^4 \geq 2(a^3+b^3+c^3)$. The equality occurs if and only if $a=b=c=2$.

Example 6. Let $a,b,c \geq -\frac{3}{4}$ and $a+b+c=1$. Prove that $\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1} \leq \frac{9}{10}$.

Proof. The inequality has form $f(a)+f(b)+f(c) \leq \frac{9}{10}$, where $f(x)=\frac{x}{x^2+1}$ with $x \in [-\frac{3}{4}; \frac{5}{2}]$. The tangent equation at the point $M(\frac{1}{3}; \frac{3}{10})$ has form $y=\frac{36x+3}{50}$.

We have $\frac{36x+3}{50}-\frac{x}{x^2+1}=\frac{(3x-1)^2(4x+3)}{50(x^2+1)} \geq 0, \forall x \in [-\frac{3}{4}; \frac{5}{2}]$.

This implies that $\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\le\frac{36(a+b+c)+9}{50}=\frac{9}{10}$. The proof is finished.

Example 7. Let acute triangle ABC. Find the maximum value of the expression

$$F=\sin A.\sin^2 B.\sin^3 C.$$

Proof. We have $\ln F=\ln\sin A+2\ln\sin B+3\ln\sin C$. Then consider the function $f(x)=\ln\sin x, x\in\left(0,\frac{\pi}{2}\right)$. It follows $f'(x)=\cot x$ and $f''(x)=-\frac{1}{\sin^2 x}, \quad \forall x\in\left(0,\frac{\pi}{2}\right)$. By applying the tagent inequality with ΔMNP , we obtain

$$f(A)\le f'(M)(A-M)+f(M)=(A-M)\cot M+\ln\sin M$$
$$f(B)\le f'(M)(B-N)+f(N)=(B-N)\cot N+\ln\sin N$$
$$f(C)\le f'(P)(C-P)+f(P)=(C-P)\cot P+\ln\sin P.$$

Thus, $\tan M.f(A)+\tan N.f(B)+\tan P.f(C)\ge\tan M\ln\sin M+\tan N\ln\sin N+\tan P\ln\sin P.$

We now choose three angles M,N,P such that

$$\frac{\tan M}{1}+\frac{\tan N}{2}+\frac{\tan P}{3}=k\Leftrightarrow\tan M=k;\tan N=2k;\tan P=3k.$$

On the orthe hand, we have $\tan M+\tan N+\tan P=\tan M.\tan N.\tan P.$

Therefore, $6k=6k^3\Leftrightarrow k=1\Rightarrow\sin M=\frac{\tan M}{\sqrt{1+\tan^2 M}}=\frac{1}{\sqrt{2}}$ and $\sin N=\frac{2}{\sqrt{5}};\sin P=\frac{3}{\sqrt{10}}$.

This implies that $f(A)+f(B)+f(C)\le\ln\frac{1}{\sqrt{2}}+2\ln\frac{2}{\sqrt{5}}+3\ln\frac{3}{\sqrt{10}}=\ln\frac{27\sqrt{5}}{125}$. So, $F\le\frac{27\sqrt{5}}{125}$. The equality occurs if and only if $A=M;B=N;C=P$.

Remark 1. Above method may apply to the following general problem

Problem 1 *Let acute triangle* ABC. *Find the maximum value of the expression* $F=\sin^m A.\sin^n B.\sin^p C,$ where m,n,p are positive real numbers.

Example 8. Let acute triangle ABC. Find the minimum value of the expression

$$F=\tan A+2\tan B+3\tan C.$$

Proof. Consider the function $f(x)=\tan x$,with $x\in\left(0,\frac{\pi}{2}\right)$. Then

$$f'(x)=1+\tan^2 x\Rightarrow f''(x)=2\tan x(1+\tan^2 x)>0,\quad x\in\left(0,\frac{\pi}{2}\right).$$

By applying the tagent inequality with acute triangle MNP, we obtain

$$f(A)\ge f'(M)(A-M)+f(M)=\frac{1}{\cos^2 M}(A-M)+\tan M\Leftrightarrow\cos^2 M.f(A)\ge\frac{1}{2}\sin 2M$$

Similarly, we also have $\cos^2 N.f(B)\ge\frac{1}{2}\sin 2N$ and $\cos^2 P.f(C)\ge\frac{1}{2}\sin 2P.$

Therefore, $\cos^2 M.f(A)+\cos^2 N.f(B)+\cos^2 P.f(C)\ge\frac{\sin 2M+\sin 2N+\sin 2P}{2}.$

We choose the angle M,N,P such that $\cos M=k;\cos N=\sqrt{2}k;\cos P=\sqrt{3}k.$ On the other hand, M,N,P are three angles of triangle, it follows $\cos^2 M+\cos^2 N+\cos^2 P+2\cos M.\cos N.\cos P=1$. Thus, $(1+\sqrt{2}+\sqrt{3})k+2\sqrt{6}k^3=1.$

This implies that k is positive solution of the following equation $2\sqrt{6}x^3+\left(1+\sqrt{2}+\sqrt{3}\right)x-1=0.$ Then we have

$$\sin 2M = 2\sqrt{1+\cos^2 M}.\cos M = 2k\sqrt{1-k^2}$$
$$\sin 2N = 2\sqrt{1+\cos^2 M}.\cos N = 2k\sqrt{2(1-2k^2)}$$
$$\sin 2P = 2\sqrt{1+\cos^2 M}.\cos P = 2k\sqrt{3(1-3k^2)},$$

and so $F \geq \dfrac{\sin 2M+\sin 2N+\sin 2P}{2k^2} = \dfrac{\sqrt{1-k^2}+\sqrt{2(1-2k^2)}+\sqrt{3(1-3k^2)}}{k}.$

This implies that the minimum value of $F = \dfrac{\sqrt{1-k^2}+\sqrt{2(1-2k^2)}+\sqrt{3(1-3k^2)}}{k}$. The equality occurs if and only if $A=M; B=N; C=P$. With M,N,P are three angles of acute triangle defined by $\cos M = k; \cos N = \sqrt{2}k; \cos P = \sqrt{3}k,$ where k is unique positive solution of equation (1).

Remark 2. Above method may apply to the following general problem.

Problem 2. *Let acute triangle ABC. Find the minimum value of the expression* $F = m\tan A + n\tan B + p\tan C,$ where m,n,p are positive real numbers and A,B,C are three angles of acute triangle.

CONCLUSION

When the teaching mathematical concepts, we should pay attention to the nature of that concept and exploit nature of the concept. In the process of teaching, we can suggest some thinking developments from a mathematical concepts to students explore and research. In vietnam, students' competency assessment is limited. This paper present a method to assess students' mathematical competence in a specific mathematics knowledge. based on the results, teachers and students could improve their teaching and learning.

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