ORIGINAL PAPER

THE NUMERICAL SOLUTION OF THE GENERAL ELLIPTIC MONGE-AMPERE TYPE EQUATION IN TWO DIMENSION : A FINITE DIFFERENCE APPROACH, IN *PDE'S*

PRAMOD KUMAR PANDEY[1]

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Abstract: *The aim of this paper is to describe an efficient numerical method for solving general elliptic Monge- Ampere type boundary value problems in two dimensions subject to the Dirichlet boundary conditions. The order of the propose method is quadratic. We have considered model linear and nonlinear problems and solved them to establish the efficiency and accuracy of the propose method.*

Keywords: *Boundary value problems, Finite difference method, Monge-Ampere equation, Quadratic order method.*

2010 for the AMS Sub. Classification: *65N06.*

1. INTRODUCTION

In this work, we consider a Monge-Ampere type boundary value problem in partial differential equations of the form

$$\frac{\partial^2 u(x,y)}{\partial x^2}.\frac{\partial^2 u(x,y)}{\partial y^2}-\left(\frac{\partial^2 u(x,y)}{\partial x\partial y}\right)^2=f\left(x,y,u(x,y),u_x(x,y),u_y(x,y)\right),\quad for\ \ (x,y)\in\Omega \tag{1}$$

where $\Omega=\{(x,y): a\le x,y\le b\}$. Let us assume that function $u(x,y)$ is convex. So equation (1) is an elliptic equation and possesses unique and numerically stable solution [1]. We consider Dirichlet boundary conditions with boundary $\partial\Omega$,

$$u(x,y)=g(x,y)\ ,\ \text{ for } (x,y) \text{ on } \partial\Omega. \tag{2}$$

For a convex function $u(x,y)$, eq. (1) is non-linear elliptic partial differential equation. The application of this class of equations appears in dynamic meteorology, elasticity, geometry and optimal transportation [2], theory of viscosity [3] and many more areas of mathematics. The increasing application of this class of PDE has generated interest in the numerical solution of the problem in last decades.

[1] Dyal Singh College (University of Delhi), Department of Mathematics, 110003 New Delhi, India.
E-mail: pramod_10p@hotmail.com.

In specific problems (1), there are many different methods and approaches such as Lagrangian [4], least squares [5], finite difference [1,6], finite element [7,8] that are used to derive the approximate solutions.

In this article we shall develop finite difference a popular method in science and engineering for solving problems (1) numerically. The order of accuracy for the propose method is at least quadratic. Numerical results validate the effectiveness and accuracy of the method in model problems. A series of papers have recently appeared in literature on numerical solution of Monge-Ampere elliptic partial differential equations, many of them are excellent work. But to best of our knowledge numerical method for the solution of problems (1) has been not discussed in literature so for.

We have presented our work in this article as follows. In the next section we will present finite difference method and in Section 3 its derivation. In Section 4, we have discussed local truncation error in propose method and the applications of the proposed method to the model problems and illustrative numerical results have been produced to show the efficiency in Section 5. Discussion and conclusion on the performance of the method are presented in Section 6.

2. THE FINITE DIFFERENCE METHOD

Consider the square domain $\Omega = [a,b] \times [a,b]$ for the solution of problem (1). Let $h = \frac{b-a}{(N+1)}$ be the uniform mesh size in the x and y directions of the Cartesian coordinate system parallel to coordinate axes. Generate mesh points $(x_i, y_j), x_i = a \pm i.h, i = 0,1,2,..,N+1$ and $y_j = a \pm j.h, j = 0,1,2,..,N+1$. Let denote the interior central mesh point (x_i, y_j) by (i,j). Consider other mesh points $(i \pm 1, j)$, $(i, j \pm 1)$ and $(i \pm 1, j \pm 1$neighbouring to the central mesh point i,j. These nine points together constitute a compact cell. So using these notations, we can rewrite problem (1) at mesh points (i,j) as follows,

$$u_{xxi,j}u_{yyi,j} - u_{xyi,j}^2 = f(x_i, y_j, u_{i,j}, u_{xi,j}, u_{yi,j}) \tag{3}$$

Here after let further simplify the notation and denote $f(x_i, y_j, u_{i,j}, u_{xi,j}, u_{yi,j})$ as $f_{i,j}$. To discretize problem (3) at mesh point (i,j) let define following approximations,

$$\bar{u}_{xi,j} = \frac{u_{i+1,j} - u_{i-1,j}}{2h} \tag{4}$$

$$\bar{u}_{yi,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2h} \tag{5}$$

and define following function

$$\bar{f}_{i,j} = f(x_i, y_j, u_{i,j}, \bar{u}_{xi,j}, \bar{u}_{yi,j}) \tag{6}$$

Following the ideas in [1], we propose a nine points second order finite difference method for the problem (3) as,

$$4(a_1 - u_{i,j})(a_2 - u_{i,j}) - \frac{1}{4}(a_3 - a_4)^2 = h^4 \bar{f}_{i,j} \tag{7}$$

where

$$a_1 = \frac{u_{i+1,j} + u_{i-1,j}}{2} \quad , \quad a_2 = \frac{u_{i,j+1} + u_{i,j-1}}{2}$$

$$a_3 = \frac{u_{i+1,j+1} + u_{i-1,j-1}}{2} \quad , \quad a_4 = \frac{u_{i-1,j+1} + u_{i+1,j-1}}{2}$$

3. DERIVATION OF THE METHOD

Consider equation (4), expand each term on right side in Taylor's series about mesh point (i, j), we will obtained

$$\bar{u}_{xi,j} = u_{xi,j} + \frac{h^2}{6}\frac{\partial^3 u(x_i, y_j)}{\partial x^3} + O(h^4) \tag{8}$$

Similarly from equation (5), we have

$$\bar{u}_{yi,j} = u_{yi,j} + \frac{h^2}{6}\frac{\partial^3 u(x_i, y_j)}{\partial y^3} + O(h^4) \tag{9}$$

Thus from equation (8) and (9), $\bar{u}_{xi,j}$ and $\bar{u}_{yi,j}$ respectively provides $O(h^2)$ approximation for $u_{xi,j}$ and $u_{yi,j}$.

Finally from (6), we will obtained

$$\begin{aligned} \bar{f}_{i,j} &= f\left(x_i, y_j, u_{i,j}, u_{xi,j} + O(h^2), u_{yi,j} + O(h^2)\right) \\ &= f\left(x_i, y_j, u_{i.j}, u_{xi,j}, u_{yi,j}\right) + O(h^2) \end{aligned} \tag{10}$$

Thus from (10), we conclude that $\bar{f}_{i,j}$ provides $O(h^2)$ approximation for $f_{i,j}$.

So from (7) and (10), we have

$$4\left(a_1 - u_{i,j}\right)\left(a_2 - u_{i,j}\right) - \frac{1}{4}(a_3 - a_4)^2 = h^4 f_{i,j} + O(h^6) \tag{11}$$

Neglecting the terms with h^6 and higher we have our propose difference method (7) for numerical solution of problem (1). The equation (7) at mesh point (i, j) defines a nonlinear equation. It means method (7) defines a system of nonlinear equations of $N \times N$ in [a, b]. Thus, the method consists in finding an approximation $u_{i,j}$ for the theoretical solution $u\left(x_i, y_j\right)$, $i, j = 1(1)N$ of the problem (1) by solving the system $N \times N$ non linear equations (7) in $u_{i,j}$.

4. LOCAL TRUNCATION ERROR

In this section, we consider the local truncation error associated to the proposed difference method (7). Let the local truncation error in (7) defined as in be $T_{i,j}$ at mesh point $(i, j), i, j = 1,2, \dots\dots, N$,

$$T_{i,j} = \frac{16(a_1 - u_{i,j})(a_2 - u_{i,j}) - (a_3 - a_4)^2}{4h^4} - \bar{f}_{i,j} \tag{12}$$

Write each term on right side of the equation (12) in Taylor series about mesh point (i, j) and simplify, we have

$$\begin{aligned} T_{i,j} = u_{xxi,j}u_{yyi,j} + \frac{h^2}{12}\left(u_{xxxxi,j}u_{yyi,j} + u_{xxi,j}u_{yyyyi,j}\right) + \cdots \\ - \left(u_{xyi,j} + \frac{1}{3}h^2\left(u_{xxxyi,j} + u_{xyyyi,j}\right) + \cdots\right)^2 - \bar{f}_{i,j} \end{aligned} \tag{13}$$

By application of (3) and (10) in (13), we have

$$T_{i,j} = \frac{h^2}{12}\left(u_{xxxxi,j}u_{yyi,j} + u_{xxi,j}u_{yyyyi,j} - 4u_{xyi,j}\left(u_{xxxyi,j} + u_{xyyyi,j}\right)\right) - O(h^2)$$

Thus we conclude that the local truncation error in propose method (7) is of $O(h^2)$.

5. NUMERICAL EXPERIMENTS

In this section, we have applied the proposed method (7) to solve numerically three different model problems. We have used Newton- Raphson method to solve the system of nonlinear linear equations arises from equation (7). All computations were performed on a Windows 2007 Ultimate operating system in the GNU FORTRAN environment version 99 compiler (2.95 of gcc) on Intel Core i3-2330M, 2.20 Ghz PC. Let $u_{i,j}$, the numerical value calculated by formulae (7), an approximate value of the theoretical solution $u(x, y)$ at the mesh point $(x, y) = (x_i, y_j)$. The maximum absolute error

$$MAE(\mathrm{u}) = \max_{1 \le i,j \le N} \left|u(x_i, y_j) - u_{i,j}\right|$$

are shown in Tables 1-3, for different value of h, the mesh size. The stopping condition for iteration was either error of order 10^{-8} or number of iterations 10^3.

Example 1. Consider a nonlinear problem which, when solving consists of

$$\frac{\partial^2 u(x,y)}{\partial x^2} . \frac{\partial^2 u(x,y)}{\partial y^2} - \left(\frac{\partial^2 u(x,y)}{\partial x \partial y}\right)^2 = \left(u(x,y)\right)^2 + \left(\frac{\partial u(x,y)}{\partial x}\right)^2 + \left(\frac{\partial u(x,y)}{\partial y}\right)^2, \quad \text{and} \quad (x,y) \in \Omega .$$

with the boundary conditions $u(x, y)$ on all sides of unit square Ω. The maximum absolute error computed in exact solution $u(x, y) = e^{(\frac{x^2+y^2}{2})}$ and presented in Table 1.

Example 2. Consider a nonlinear problem which, when solving consists of

$$\frac{\partial^2 u(x,y)}{\partial x^2}.\frac{\partial^2 u(x,y)}{\partial y^2} - \left(\frac{\partial^2 u(x,y)}{\partial x \partial y}\right)^2 = \frac{\partial u(x,y)}{\partial x}.\frac{\partial u(x,y)}{\partial y} + f(x,y), \text{ and } (x,y) \in \Omega\,.$$

with the boundary conditions $u(x,y)$ on all sides of unit square Ω. $f(x,y)$ is given such that the exact solution is $u(x,y) = \frac{2\sqrt{2}}{3}(x^2+y^2)^{\frac{3}{4}}$. The maximum absolute error computed in considered exact solution and presented in Table 2.

Example 3. Consider a nonlinear problem which, when solving consists of

$$\frac{\partial^2 u(x,y)}{\partial x^2}.\frac{\partial^2 u(x,y)}{\partial y^2} - \left(\frac{\partial^2 u(x,y)}{\partial x \partial y}\right)^2 = \frac{1}{2xy}.\frac{\partial u(x,y)}{\partial x}.\frac{\partial u(x,y)}{\partial y}, \text{ and } (x,y) \in \Omega\,.$$

with the boundary conditions $u(x,y)$ on all sides of unit square Ω. The maximum absolute error computed in exact solution $u(x,y) = \frac{2\sqrt{2}}{3}(x^2+y^2)^{\frac{3}{4}}$ and presented in Table 3.

Table 1.Maximum absolute error $|u(x_i, y_j)) - u_{i,j}|$ in example 1 .

	N				
	8	*16*	*32*	*64*	*128*
MAE	.27382374(-3)	.67949295(-4)	.12516975(-4)	.47683716(-6)	.23841858(-6)

Table 2. Maximum absolute error $|u(x_i, y_j)) - u_{i,j}|$ in example 2 .

	N				
	8	*16*	*32*	*64*	*128*
MAE	.22431633(-2)	.78406668(-3)	.27571924(-3)	.97263743 (-4)	.34355708 (-4)

Table 3. Maximum absolute error $|u(x_i, y_j)) - u_{i,j}|$ in example 3 .

	N				
	8	*16*	*32*	*64*	*128*
MAE	.35331196(-2)	.12530528(-2)	.44314962(-3)	.15668073(-3)	.15668073(-3)

CONCLUSION

A finite difference method for numerical solution is presented for numerical solution of nonlinear Monge-Ampere elliptic PDEs. It follows from derivation and discussion that the proposed method (7) is of at least quadratic order which is well evident in computational results. We can claim, in general that our method is simple, convergent and accurate finite difference method. Numerical results show that our method generate stable numerical results except in example 1. Though we have developed method on square domain and used equally spaced grid mesh size, it has good potential for efficient application to many problems on different geometries; work in this specific direction is in progress.

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